Midterm Exam I
Math 234 – Fall 2013 – Lecture 5

Student’s Name:

TA’s Name: ___________________________ Section Number: ____________

Instructions

• Please do not open the exam until you are told to do so.
• Please silence your cell phones.
• You are not allowed to use calculators and notes.
• Please show your work.
• Please raise your hand if you have a question.
• You are not allowed to leave until the 50-minute period is over.
• Please stop working on the exam when you are told to do so.
• May the Force be with you.

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| TA/Section | 5 |
| extra credits | ( /10) |
| Total       | /105 |
1. Let

\[ f(x, y) = \begin{cases} \frac{x^2 y^2 - x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ c, & (x, y) = (0, 0). \end{cases} \]

(a) Find \( \lim_{(x,y) \to (0,0)} f(x,y) \) or show that it does not exist.

(b) Is there a value of \( c \) that makes \( f \) continuous at the origin? Why or why not?
2. Let
\[ g(x, y) = \frac{(x - y)^2}{2} + 1. \]

- Find and draw the level curves (i.e. contours) \( g^{-1}(1) \) and \( g^{-1}(3) \).
- Find and draw \( \nabla g \) at the points (3, 1) and (-2, 0).

*Note:* Your drawings must be as accurate as possible.
3. Consider the equation

\[ xy^2z + (x + 4y + 2z)^3 = 2. \]

(a) This equation defines \( z \) implicitly as a function of \( x \) and \( y \). Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

(b) The given equation also describes a level surface of a function of \( x, y, z \). Find the equation of the normal line to this surface at the point \((2, -1, 1)\).
4. A bug is crawling on a plane whose temperature at the point \((x, y)\) is given by

\[ T(x, y) = 75 + x^2y. \]

(a) Find the rate of change of \(T\) if the bug moves from the point \((2, -1)\) in the
direction of the vector \(\hat{i} - 2\hat{j}\).

(b) In which direction should the bug moves from the point \((2, -1)\) in order to cool
off as quickly as possible? What is the rate of change of \(T\) in that direction?

*Note:* A direction must be a unit vector.
5. Let

\[ h(x, y) = xe^{\sin y}. \]

(a) Find \( h_x, h_y, h_{xx}, h_{yy}, h_{xy}, h_{yx} \).

(b) Find the linear (i.e. first-order) approximation of \( h_y \) near \((3, \pi)\).

Notes:
- Be careful! We are interested in the linear approximation of \( h_y \), not of \( h \).
- Your answer should be a function of \( x \) and \( y \).

(c) Use your result in Part (b) to approximate \( h_y \) at \((\pi, 3)\).

Notes:
- Your answer should be a number. Use \( \pi \approx 3.14 \).
- After the exam, you should use your calculator to check how close your approximation is to the exact value.
Extra credits: In the ball-and-stick model, a methane molecule (CH₄) is said to possess tetrahedral shape. This means that the four hydrogen atoms coincide with the four vertices of a regular tetrahedron, with the carbon atom at the center. (A regular tetrahedron is a solid with four equilateral-triangle faces as shown below. Put another way, the distance between any two of the four vertices of a regular tetrahedron is a constant.) In your chemistry class, you learn that the angle between two arms of this molecule is about 109.5°. (An arm is a stick joining the carbon atom and a hydrogen atom.) Our goal is to compute this angle using our knowledge of vectors.

Credits to Ben Mills (Wikipedia) for the figure on the left.

(a) First we show that a regular tetrahedron can be packed into a cube as follows. Imagine a cube whose eight vertices are at the points (±1, ±1, ±1) and whose center is at the origin. Consider four vertices of this cube:

\[ P(1, 1, 1), \quad Q(1, -1, -1), \quad R(-1, 1, -1), \quad S(-1, -1, 1). \]

Show that the distance between any two of these four points is a constant.

Note: This entails finding six distances for six possible pairings of four points.
(b) Your result in Part (a) implies that a regular tetrahedron can be packed into a cube; the four vertices of the regular tetrahedron (i.e. the four hydrogen atoms) are at $P, Q, R, S$. By symmetry, the center of the regular tetrahedron (i.e. the carbon atom) must be at the origin. Use this information to find the angle between two arms of a methane molecule.

*Note:* Your answer will be an arccos of a number. After the exam, you should use your calculator to check whether this really is $109.5^\circ$.

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**FOXTROT**

**How was the big math test?**

**Outstanding.**

**Shades of Fall '97**

With daring reminders of finals '01.

**Imagine the playfulness of a mid-term '99**

Coupled with the difficulty of a late '97 or '98.

**You know, there's a thin line between connoisseur**

And "nut case." Jason.

**I still have an old '98 upstairs.**

I really should take it again.

Credits to Bill Amend.