1. (3 points) Let \( p(\hat{r}) = p_0 - \rho g (\hat{z} \cdot \hat{r}) \), where \( p_0 \), \( \rho \) and \( g \) are constants. Compute \( \oint_S p \hat{n} \, dS \), where \( S \) is a closed surface and \( \hat{n} \) is the outside unit normal to \( S \).

\[
\begin{align*}
p &= p_0 - \rho g (\hat{z} \cdot \hat{r}) = p_0 - \rho g z \\
\nabla p &= \frac{\partial p}{\partial x} \hat{x} + \frac{\partial p}{\partial y} \hat{y} + \frac{\partial p}{\partial z} \hat{z} = -\rho g \hat{z} \\
\oint_S p \hat{n} \, dS &= \int_V \nabla p \, dV = \int_V -\rho g \hat{z} \, dV \\
&= -\rho g \int_V \hat{z} \, dV \\
&= -\rho g \text{vol}(V) \hat{z}
\end{align*}
\]

2. (2 points) Complete the following identities:

(a) \( \int_S (\nabla \times \bar{F}) \cdot \hat{n} \, dS = \oint_C \bar{F} \cdot \hat{r} \, d\bar{r} \), \( C = \partial S \)

(b) \( \int_V \frac{\partial f}{\partial x_j} \, dV = \oint_S \nabla f \cdot \hat{n}_j \, dS \), \( C = \partial V \)

and \( \hat{n}_j \) is the \( j \)-th cartesian coordinate of the outward unit normal to \( S \).
3. (5 points) Let \( \mathbf{v}(r) = \mathbf{r} \).

(a) Compute the flux of \( \mathbf{v} \) through the surface of a sphere of radius \( R \) centered at the origin in two ways.

(b) Compute the flux of \( \mathbf{v} \) through the surface of a sphere of radius \( R \) centered at \( \mathbf{r}_0 \).

\[ \text{a) 1st way: By direct computation} \]
\[
\mathbf{\nabla}_S \cdot \mathbf{n} \, dS = \oint_{S_R} \mathbf{r} \cdot \mathbf{n} \, dS = \oint_{S_R} r \, (\mathbf{r} \cdot \mathbf{n}) \, dS
\]
\[
= \oint_{S_R} r \, dS = \oint_{S_R} R \, dS
\]
\[
= R \oint_{S_R} dS = R \cdot 4\pi R^2 = 4\pi R^2
\]

where \( \mathbf{r} = \mathbf{r}'(r) = \mathbf{r} \) and \( r = 1 \mathbf{r}' = R \), since \( \mathbf{r}' \) lies on the sphere \( S_R \) of radius \( R \) and centered at \( \mathbf{0} \).

\[ \text{2nd way: Using Gauss' (divergence) theorem} \]
\[
\mathbf{\nabla} \cdot \mathbf{v} = 1 \implies \mathbf{\nabla} \cdot \mathbf{r} = 3 \]
\[
\mathbf{\nabla}_S \cdot \mathbf{n} \, dS = \mathbf{\nabla}_R \mathbf{v} \cdot \mathbf{n} \, dV = \mathbf{\nabla}_R \mathbf{v} \cdot dV = 3 \mathbf{\nabla}_R \mathbf{v} \cdot dV
\]
\[
= 3 \cdot \frac{4\pi R^3}{3} = 4\pi R^3
\]

b) The 2nd way above still applies so we still get \( \oint_{S_{R,0}} \mathbf{v} \cdot \mathbf{n} \, dS = 4\pi R^3 \)