Answers

2. \(-\sqrt{2}\pi\)
3. 0
4. (a) \(\vec{B} = \frac{1}{\rho} \hat{\rho}\), defined for \(\rho \neq 0\), i.e. away from z-axis,
   (b) \(\vec{\nabla} \times \vec{B} = 0\), (c) i. \(2\pi\), ii. 0, iii. \(2\pi\)
5. \(4\pi/3\)
6. \(3L^3\)
7. (a) defined for \(\vec{r} \neq 0\), (b) \(\vec{\nabla} \cdot \vec{v} = 0\), (c) i. \(4\pi\), ii. 0, iii. \(4\pi\)
8. I don’t!
9. (a) \(|z| = e^3\), \(\text{arg}(z) = -3\pi/4\), (b) \(\ln i = i\pi/2\), \(i^i = e^{-\pi/2}\)
10. (a) \(\pm \sqrt{2}/2 \pm i\sqrt{2}/2\), (b) \(\pm \sqrt{3+i}, -2i\), (c) \(z_k = e^{i\frac{10}{21}(2k+1)}\pi\), \(k = 0, 1, \ldots, 20\)
   (d) 2, \(\pm i\)
11. (a) \(\sum_{n=0}^\infty (-1)^n(n+1)z^n\), \(R = 1\), (b) \(\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}(z-2)^n\), \(R = 2\)
   (c) \(\sum_{n=0}^\infty \frac{(-1)^n}{(1+i)^n}(z-i)^n\), \(R = 1\), (d) \(\sum_{n=0}^\infty \frac{(-1)^n}{(1+2i)^n+1}(z-(1+2i))^{n+1}\), \(R = 5\)
12. (a) no, (b) no, (c) yes, (d) no
13. (a) \(v(x, y) = \arctan(y/x) (+C)\), (b) \(v(x, y) = \cosh x \sin y (+C)\)
Hints

1. Start with
\[ \nabla \cdot (\vec{v} \times \vec{w}) = (\varepsilon_{klm} \varepsilon_k v_l w_m) = \partial_i (\varepsilon_{klm} v_l w_m) (\varepsilon_i \cdot \varepsilon_k) \]
and then follow your nose, having in mind that \( \varepsilon_{klm} \) is a number, while \( v_l \) and \( w_m \) are functions.

2. You can parametrize the ellipse as \( \vec{r}(t) \equiv (\sqrt{2} \cos t, \sin t), \) \( 0 \leq t \leq 2\pi \). Then the line integral is equal to
\[ \int_0^{2\pi} (\sqrt{2} \cos t + 3 \sin t, 2\sqrt{2} \cos t - \sin^2 t) \cdot (-\sqrt{2} \sin t, \cos t) \, dt; \]
first do the dot product and then compute the integral. Alternatively you can use Green’s theorem (84).

3. You can assume a parametrization \( \vec{r}(t), t_0 \leq t \leq t_f \), for \( C \) and then compute \( \oint_C \vec{r} \cdot d\vec{r} = \int_{t_0}^{t_f} r^\alpha \hat{r} \cdot \frac{d\vec{r}}{dt} \, dt \), using that \( \hat{r} \cdot \frac{d\vec{r}}{dt} = \frac{dr}{dt} \). Alternatively you can find \( f \) such that \( \nabla f = r^\alpha \hat{r} \). Or you can just use Stokes’ theorem (86), after computing \( \nabla \times (r^\alpha \hat{r}) \). (Hey, that’s three ways!)

4. (b) Since \( \vec{B} = \frac{1}{\rho} \hat{\phi} \), you can write \( \nabla \) in cylindrical coordinates and then let it act on \( \frac{1}{\rho} \hat{\phi} \), or you can first compute \( \nabla \hat{\phi} \) and then use the product rule (67), as done in the solution of quiz 7. Alternatively you can show that \( \vec{B} = -\frac{y}{x^2+y^2} \hat{e}_x + \frac{x}{x^2+y^2} \hat{e}_y \) and then compute the curl in cartesian coordinates.

(c) i. You can calculate \( \oint_C \vec{B} \cdot d\vec{r} \), by parametrizing the circle \( C \). See solution of quiz 4M. You cannot use Stokes’ theorem since \( C \) goes around the z-axis, where \( \vec{B} \) is undefined.
ii. Here you can use Stokes’ theorem.
iii. You cannot use Stokes’ theorem directly, since \( C \) goes around the z-axis. Introduce a small circle \( C_R \) centered at the origin on a plane perpendicular to \( \hat{e}_z \) as in i. Then you can find a surface whose boundary is \( C - C_R \). For that surface you can use Stokes’ theorem, so \( \oint_C \vec{B} \cdot d\vec{r} - \oint_{C_R} \vec{B} \cdot d\vec{r} = 0 \).

5. Use Gauss’ theorem (111).
6. We have that \( \oint_S \vec{v} \cdot \vec{n} \, dS = \oint_S \vec{r} \cdot \vec{n} \, dS = \sum_{j=1}^{6} \int_{F_j} \vec{r} \cdot \vec{n} \, dS \), where \( F_j \) are the faces of the cube. Each integral in the sum can be calculated by parametrizing \( F_j \). For example if \( F_1 \) is the face of the cube that lies on the \( xy \)-plane, then we can parametrize it by \( \vec{r}(x, y) = x \vec{e}_x + y \vec{e}_y \), \( 0 \leq x, y \leq L \). Also notice that in this case \( \vec{n} = \vec{e}_z \). We find \( \int_{F_1} \vec{r} \cdot \vec{n} \, dS = 0 \). Similarly the faces lying on the \( xz \) and \( yz \)-planes will have zero contribution, while the remaining three faces will contribute \( L^2 \) each. Alternatively you can use Gauss’ theorem.

7. (b) Use the product rule (66).
   (c) i. ii. See solution of quiz 8M.
   iii. You cannot apply Gauss’ theorem directly, since \( S \) encloses the origin where \( \vec{v} \) is undefined, so use again the isolate the singularity technique: introduce a small sphere \( S_\varepsilon \) of radius \( \varepsilon \) centered at 0. Then you have created a volume whose boundary is \( S - S_\varepsilon \) and for which you can apply Gauss’ theorem, since it does not include the origin. Therefore \( \oint_S \vec{v} \cdot \vec{n} \, dS - \oint_{S_\varepsilon} \vec{v} \cdot \vec{n} \, dS = 0 \).

8. In Stokes’ theorem \( S \) is surface with boundary, its boundary is \( C \). In Gauss’ theorem \( S \) is a closed surface (no boundary), so a consecutive application of these two theorems does not make sense.

9. (b) Use formulas (46) and (47).

10. (a), (b) Use formula (51).
    (c) Using the polar representation \( z = re^{i\theta} \) we have
        \[
        z^{2.1} + 1 = 0 \iff z^{\frac{21}{10}} = -1 \iff e^{\frac{21}{10} \ln z} = e^{i\pi} \iff e^{\frac{21}{10} (\ln r + i\theta)} = e^{i\pi} \]
        \[
        \iff r^{\frac{21}{10}} e^{\frac{21}{10} i\theta} = e^{i\pi} \]
        and the last equation implies \( r^{\frac{21}{10}} = 1 \) and \( \frac{21}{10} \theta = \pi + 2k\pi, k = 0, \pm 1, \ldots \).
        We conclude \( z = z_k = e^{i\frac{10}{11} (2k+1)\pi} \), but only \( k = 0, 1, \ldots, 20 \) give us distinct roots.
    (d) Factor the polynomial.

11. (a) You can use (17), but you need to compute a bunch of derivatives. It is easier to start with \( \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \), which is true for \( |z| < 1 \) and differentiate both sides. Remember the derivative of the infinite sum
will be the infinite sum of the derivatives and the radius of convergence is equal to that of the original series.

(b), (c) See (42).

(d) Use (42) with $a = 1 + 2i$ and then take antiderivatives of both sides.

12. You can use the definition of the complex derivative (7). (a) is done this way in the notes (see (56)). Alternatively all of them can be answered easily using the Cauchy-Riemann equations (55).

13. If $f$ is complex differentiable, then $u$ and $v$ have to satisfy the Cauchy-Riemann equations. For example for (b) these give $\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y} = \sinh x \sin y \ (\ast)$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \cosh x \cos y \ (\ast)$ (remember $(\sinh)' = \cosh$ and $(\cosh)' = \sinh$). Integrating $(\ast)$ with respect to $x$ yields $v(x, y) = \cosh x \sin y + h(y)$, where $h$ is some arbitrary function of $y$. Differentiating this one with respect to $y$ we get $\frac{\partial v}{\partial y} = \cosh x \cos y + h'(y)$. Comparing this with $(\ast)$ we see that $h'(y) = 0$, so $v(x, y) = \cosh x \sin y + h(y) = \cosh x \sin y + C$. 