PRACTICE EXAM

1. (Limits) Use the $\varepsilon - \delta$ definition of a limit to show that \( \lim_{x \to 3} 5x - 7 = 8. \)

   **Solution:** Given \( \varepsilon > 0, \) let \( \delta = \frac{\varepsilon}{5}. \) Then if \( 0 < |x - 3| < \delta, \) we have that
   \[
   |(5x - 7) - 8| = |5x - 15| = 5|x - 3| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.
   \]
   Therefore, by the definition of limits, we have that \( \lim_{x \to 3} 5x - 7 = 8. \)

2. (Exponents) Compute \( \frac{4^{1000} \cdot (-\frac{1}{2})^{2015}}{(\frac{1}{2})^{20}} \)

   **Solution:**
   \[
   \frac{4^{1000} \cdot (-\frac{1}{2})^{2015}}{(\frac{1}{2})^{20}} = \frac{2^{2000} \cdot 2^{-2015} \cdot (-1)^{2015}}{2^{-20}} = \frac{2^{-15}}{2^{-20}} = 2^5 = 32.
   \]

3. (Inequalities) Find all \( x \) such that \( \frac{1}{2x+1} > \frac{3}{4x+5}. \)

   **Solution:**
   \[
   \frac{1}{2x + 1} - \frac{3}{4x + 5} > 0,
   \]
   \[
   \frac{-2(x - 1)}{(2x + 1)(4x + 5)} > 0,
   \]
   \[
   x \in (-\infty, \frac{5}{4}) \cup (-\frac{1}{2}, 1)
   \]

4. (Inverse Functions/Quadratics) Let \( g(x) = x^2 - 15x + 100. \) Find \( g^{-1}(50) \) in each of the following cases if it exists.

   (a) Domain of \( g \) is \( (-\infty, 7). \)
   (b) Domain of \( g \) is \( (11, 20). \)
   (c) Domain of \( g \) is \( (-20, 20). \)

   **Solution:**
   \[
   x^2 - 15x + 100 = 50 \quad \Rightarrow \quad x = 5, 10
   \]
   (a) Only 5 belongs to the domain of \( g. \) Therefore, \( g^{-1}(50) = 5. \)
   (b) Neither 5 nor 10 belong to the domain of \( g. \) Therefore, \( g^{-1}(50) \) does not exist.
   (c) Both 5 and 10 belong to the domain of \( g. \) Therefore, the horizontal line test fails and \( g^{-1}(50) \) does not exist.
5. (Quadratics) Let \( f(x) = x^2 - 6x + 8 \).

(a) Write \( f(x) \) in the vertex form and state its vertex.
(b) Find the largest interval containing 0 such that \( f \) is invertible.
(c) Find \( f^{-1} \) for that domain.
(d) Find the domain and the range of \( f^{-1} \)

**Solution:**

(a) Complete the square to find the vertex form for \( f \):
\[
f(x) = x^2 - 6x + 8 = x^2 - 6x + 9 - 1 = (x - 3)^2 - 1.
\] So the vertex of \( f \) is \((3, -1)\).

(b) In order for \( f \) to be invertible, it should be 1-1. So we need to find the largest interval containing 0 such that \( f \) is 1-1. The largest intervals on which \( f \) is 1-1 are \([3, \infty)\) and \((\infty, 3]\). \((\infty, 3]\) is the one containing 0.

(c) 
\[
y = (x - 3)^2 - 1 \Rightarrow y + 1 = (x - 3)^2
\]
\[
\Rightarrow \pm \sqrt{y + 1} = x - 3
\]
Because the domain of \( f \) is \((\infty, 3]\), we should only consider \(-\sqrt{y + 1}\). So, \(-\sqrt{y + 1} = x - 3 \Rightarrow -\sqrt{y + 1} + 3 = x\)
So, \( f^{-1}(y) = -\sqrt{y + 1} + 3 \).

(d) The range of \( f^{-1} \) is the domain of \( f \): So it is \((\infty, 3]\) and the domain of \( f^{-1} \) is the range of \( f \): So it is \([-1, \infty]\).

6. (Absolute Values)

(a) Find all \( x \) such that \( |x - 3| + |x + 1| = 5 \).
(b) Using the geometrical interpretation of \(|\cdot|\) explain why none of your answers were in the interval \([-1, 3]\).

**Solution:**

(a) Since \(|x - a|\) represents the distance between \( a \) and \( x \), we are looking for a point such that its distance to 3 plus its distance to \(-1\) add to \(5\).

First, we know that the distance between 3 and \(-1\) is 4, then the number cannot be between \(-1\) and 3 (there, the sum is always 4).

Now, if \( x \leq -1 \) then its distance to 3 is going to be the distance it has to \(-1\) plus the remaining distance it has to 3 in other words:
\[
|x - 3| + |x + 1| = |x - 3| + |x - (-1)| = |x - (-1)| + 3 - (-1) + |x - (-1)| = 4 + 2|x - (-1)|.
\]
So we are looking for an \( x \) such that twice its distance to \(-1\) plus 4 is 5, or such that twice its distance to \(-1\) is \(5 - 4 = 1\). Therefore, the number we’re looking for has a distance of \(\frac{1}{2}\) from \(-1\) and it is less than it. It has to be \(-1 - \frac{1}{2} = -\frac{3}{2}\).

If \( x \geq 3 \) we can follow similar reasoning, so we will need a number bigger that 3 such that its distance to 3 is \(\frac{1}{2}\). This number is \(3 + \frac{1}{2} = \frac{7}{2}\).
(b) We know that the distance between 3 and −1 is 4, so $|x - 3| + |x + 1|$ cannot take the value 5 between −1 and 3, for every number in that interval, $|x - 3| + |x + 1| = 4$

7. (Distance/Lines/Quadratics) What point on the graph of $y = 2x + 1$ is closest to $(0, -4)$? Compute that distance.

**Solution:** The closest point to $(0, -4)$ on the line $y = 2x + 1$ is the one that is the intersection between $y = 2x + 1$ and a perpendicular line to it going through $(0, -4)$.

We know that the perpendicular line must have slope $-\frac{1}{2}$, so we have the equation

$$-\frac{1}{2} = \frac{y - (-4)}{x - 0}$$

solving for $y$ we got $y = \frac{-1}{2} x - 4$.

Now, to find the intersection, we set the line equations equal. Then

$$2x + 1 = \frac{-1}{2} x - 4; 2x + \frac{1}{2} x = -4 - 1; \frac{5}{2} x = -5; x = -5 \cdot \frac{2}{5} = -2.$$ 

Since $x = -2$ we have that $y = 2(-2) + 1 = -3$.

Finally, the distance between the two points is

$$\sqrt{(0 - (-2))^2 + (-4 - (-3))^2} = \sqrt{(2)^2 + (-1)^2} = \sqrt{4 + 1} = \sqrt{5}$$