(1) Let \( f(\lambda) \in F[\lambda] \) be a monic polynomial of degree \( n \geq 1 \). Let \( E \) be the splitting field of \( f(\lambda) \) over \( F \) and suppose that \( f(\lambda) \) has distinct roots in \( E \). Let \( K_1 \) and \( K_2 \) be subfields of \( E/F \) so that \( [K_1 : F][K_2 : F] = [E : F] \).

Let \( G = Gal(E/F) \), \( G_1 = Gal(E/K_1) \) and \( G_2 = Gal(E/K_2) \).

(So \( G_1 \) and \( G_2 \) are subgroups of \( G \).)

Show that
a) \( G_1 \cap G_2 = \{ \epsilon \} \)
b) \( G_1G_2 = G. \)

(2) Suppose that \( E/F \) is a finite extension of degree \( n \).

a) Let \( E'/F' \) be an extension and let \( \phi : F \to F' \) be an isomorphism. Show that the number of extensions of \( \phi \) to a homomorphism of \( E \) to \( E' \) is \( \leq n \).

b) Show that \( |Gal(E/F)| \leq n \).

(3) Let \( f(\lambda) \in \mathbb{Q}[\lambda] \) be given below. Let \( E \) be the splitting field of \( f(\lambda) \) over \( \mathbb{Q} \) and let \( G = Gal(E/\mathbb{Q}) \). Find \( E \) and find \( [E : \mathbb{Q}] \).

a) \( f(\lambda) = \lambda^5 - 2 \in \mathbb{Q}[\lambda] \)
b) \( f(\lambda) = \lambda^4 - 4\lambda^2 + 2 \in \mathbb{Q}[\lambda] \) (Hint: Show that \( E \) is generated by a single root of \( f(\lambda) \) in \( E \).)

(4) Let \( E = \mathbb{F}_2(\lambda)/(q(\lambda)) \) where \( q(\lambda) = \lambda^4 + \lambda + 1 \in \mathbb{F}_2(\lambda) \). Then
\[
E = \{ a_0 + a_1 r + a_2 r^2 + a_3 r^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{F}_2 \}
\]
where \( r = \lambda + (q(\lambda)) \). From our previous assignment we know that \( E \) is a field of order 16. Find a generator for the group \( E^\times = \{ u \in E \mid u \neq 0 \} \).

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\(^1\)The field generated by \( K_1 \) and \( K_2 \) denoted \( \langle K_1, K_2 \rangle \) is the smallest subfield of \( E \) containing \( K_1, K_2 \).

\(^2\)Recall \( G_1G_2 = \{ g_1g_2 \mid g_1 \in G_1, g_2 \in G_2 \} \). In general it is not a subgroup.