What is the inverse of a function? Technically, the inverse of a function $f(x)$ (if it even has one!) is a function $g(x)$ with the property that, for all $x$ in the domain of $f(x)$, we have $g(f(x)) = x$, and for all $y$ in the range of $f$, we have $f(g(y)) = y$. We denote the inverse of $f(x)$ by $f^{-1}(x)$.

Informally, $f^{-1}$ is the function whose inputs and outputs are the same as those of $f$, only switched. For example, if $f$ is a function that has an inverse (such a function is called invertible), then if $f(3) = 1$, we know that $f^{-1}(1) = 3$. A function is invertible if, for any value $y$ that is an output of $f$, there is only one input that gives that output, i.e. if $f$ is “one-to-one.” (Otherwise, what would $f^{-1}(y)$ be? There can only be one value for this, or else $f^{-1}$ isn’t a function!)

1. The graph of $y = f(x)$ is shown. What is $f^{-1}(2)$?

2. Find the domain and range of each function, and draw a rough graph. Then find the function’s inverse, find its domain and range, and draw this graph on the same set of axes.

(a) $f(x) = x^3 + 5$
(b) $g(x) = \sqrt{1 - x}$
(c) $h(x) = \sqrt[3]{1 - x^3}$ (What is unusual about this function and its inverse?)

3. Draw a nice graph of $y = \sin(x)$. Notice that, if we restrict the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the sine function is one-to-one (explain why). Use this to draw the graph of the inverse of $y = \sin(x)$, when the domain of $\sin(x)$ is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This inverse is called arcsin, or $\sin^{-1}$. What are its domain and range?

4. Repeat the previous problem with $y = \tan(x)$ instead of $y = \sin(x)$. This time, you choose the domain restriction by looking at the graph of the tangent function.

5. What is

(a) $\arcsin(\sin(\frac{\pi}{4}))$
(b) $\arcsin(\sin(\frac{3\pi}{4}))$
(c) $\sin(\arcsin(3/4))$
(d) $\sin(\arcsin(4/3))$

6. (a) Let $f(x) = \arcsin(\sin(x))$. What are the domain and range of $f$? Can you simplify the formula for this function?

(b) Repeat part (a) with $f(x) = \sin(\arcsin(x))$. 

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