Recall that we say a function $y = f(x)$ has a horizontal asymptote of $y = b$ if

$$\lim_{x \to \infty} f(x) = b, \text{ or } \lim_{x \to -\infty} f(x) = b$$

(so a function can have up to 2 different horizontal asymptotes... like arctan!)

Another way of saying $\lim_{x \to \infty} f(x) = b$ is to say $\lim_{x \to \infty} f(x) - b = 0$. This motivates the following definition: we say $f(x)$ has a slanted asymptote of $y = mx + b$ if

$$\lim_{x \to \infty} f(x) - (mx + b) = 0, \text{ or } \lim_{x \to -\infty} f(x) - (mx + b) = 0.$$

It’s often useful to think of this definition as saying that $f(x)$ has a slanted asymptote of $y = mx + b$ if we can write $f(x) = mx + b + \ast$, where $\ast \to 0$ as $x \to \infty$ or as $x \to -\infty$.

One situation where we can write a function this way by using long division on a rational function, like $f(x) = \frac{x^4 + x^3 - 1}{2x^4 - x + 1}$

Using long division we find that

$$f(x) = \frac{1}{2}x + \frac{1}{2} + \frac{\frac{1}{2}x^2 - \frac{3}{2}}{2x^3 - x + 1}.$$ 

Since the last term approaches 0 as $x \to \pm \infty$, we know $f(x)$ has a slanted asymptote of $y = \frac{1}{2}x + \frac{1}{2}$, which the graph approaches as $x \to \infty$ and $x \to -\infty$.

Problems – find all slanted asymptotes.

1. $f(x) = \frac{x^2 + 1}{x - 1}$
2. $f(x) = \frac{2x^3 - 4x + 5}{x^2 + 1}$
3. $f(x) = \sqrt{x^2 - 1}$. (Hint: what does the graph of $y = \sqrt{x^2}$ look like?)