Problem 5: \( \rho \frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + Q \), \( \epsilon, \rho, k_0 \) = constants.

Part (a) \( Q = 0, \ u(0) = 0, \ u(L) = T \)

\( \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2 \)

\( 0 = u(0) = c_2 \Rightarrow c_2 = 0 \)

\( T = u(L) = c_1 \ L \Rightarrow c_1 = \frac{T}{L} \)

\( \Rightarrow u = \frac{T}{L} x + 0 = \frac{T}{L} x. \)

Part (b) \( Q = 0, \ u(0) = T, \ \frac{\partial u}{\partial x} (L) = \alpha \)

\( \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u(x) = c_1 x + c_2 \)

\( T = u(0) = c_2 \Rightarrow c_2 = T \)

\( \alpha = \frac{\partial u}{\partial x} (L) = c_1 \Rightarrow c_1 = \alpha \)

Therefore \( u = \alpha x + T. \)

Part (c) \( 0 = k_0 \frac{\partial u}{\partial x^2} + Q \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{Q}{k_0} = -x^2 \)

\( \Rightarrow \frac{\partial u}{\partial x} = -\frac{x^3}{3} + c_1 \Rightarrow u = -\frac{x^4}{12} + c_1 x + c_2 \)

\( T = u(0) = c_2 \Rightarrow c_2 = T \)

\( 0 = \frac{\partial u}{\partial x} (L) = c_1 - \frac{L^3}{3} + c_1 = \Rightarrow c_1 = \frac{L^3}{3} \)

\( \Rightarrow u = -\frac{x^4}{12} + \frac{L^3}{3} x + T. \)

Part (d) \( \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2 \)

\( \alpha = \frac{\partial u}{\partial x} (L) = c_1 \Rightarrow c_1 = \alpha \)

\( 0 = \frac{\partial u}{\partial x} (0) - [u(0) - T] = c_1 - [c_2 - T] = \alpha - [c_2 - T] = \Rightarrow c_2 = \bar{T} + \alpha \)

\( \Rightarrow u = \alpha x + T + \alpha. \)
Problem 6:

Part (a) \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 \), \( U(x, 0) = f(x) \), \( \frac{\partial u}{\partial x}(0, t) = 1 \), \( \frac{\partial u}{\partial x}(L, t) = \beta \).

\( \frac{\partial^2 u}{\partial x^2} = -1 \implies \frac{\partial u}{\partial x} = -x + C_1 \implies u = -\frac{x^2}{2} + C_1 x + C_2 \)

\( 1 = \frac{\partial u}{\partial x}(0, t) = -0 + C_1 \implies C_1 = 1. \)

\( \beta = \frac{\partial u}{\partial x}(L, t) = -L + C_1 = -L + 1 \)

If \( \beta = 1 - L \), there is an equilibrium solution, which is

\( u = -\frac{x^2}{2} + x + C_2 \).

If \( \beta \neq 1 - L \), there isn't an equilibrium solution.

The difficulty is caused by the heat flow being specified at both ends and a source specified inside.

An equilibrium will exist only if these three are in balance. This balance can be mathematically verified from conservation of energy:

\[
\frac{d}{dt} \int_0^L \rho u \, dx = \int_0^L \rho \frac{\partial u}{\partial t} \, dx = \int_0^L \frac{\partial}{\partial x} \left( -\frac{x^2}{2} + x + C_2 \right) + Q \, dx \quad , \quad Q = 1
\]

\( = \phi(0) - \phi(L) + L = -\frac{\partial u}{\partial x}(0) + \frac{\partial u}{\partial x}(L) + L = -1 + \beta + L \)

If \( \beta = 1 - L \implies \) the total energy is constant

\( \implies \int_0^L f(x) = \int_0^L u(x) \, dx = \int_0^L -\frac{x^2}{2} + x + C_2 = -\frac{L^3}{6} + \frac{L^2}{2} + C_2 L \)

\( \implies C_2 = -\frac{1}{L} \left( -\frac{L^3}{6} + \frac{L^2}{2} \right) \)

\( \implies u = -\frac{x^2}{2} + x + \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L} \int_0^L f(x) \, dx \)
Part (b) \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \), \( u(x,0) = f(x), \frac{\partial u}{\partial x}(0,t) = 1, \frac{\partial u}{\partial x}(L,t) = \beta \)

\[
\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2
\]

\[4 = \frac{\partial u}{\partial x}(0) = c_1 \Rightarrow c_1 = 1\]

\[\beta = \frac{\partial u}{\partial x}(L,t) = c_1 = 1\]

If \( \beta = 1 \), there is an equilibrium solution.

\[u = x + c_2\]

and if \( \beta \neq 1 \), there are no equilibrium solutions.

To determine, we check that the total energy is constant:

\[
\frac{d}{dt} \int_0^L u(x) dx = \int_0^L \frac{\partial u}{\partial t} dx = \int_0^L \frac{\partial^2 u}{\partial x^2} dx
\]

\[
= \frac{\partial u}{\partial x}(L) - \frac{\partial u}{\partial x}(0) = 1 - \beta = 0
\]

\[\Rightarrow \int_0^L u(x) dx \text{ is constant in time}\]

\[\Rightarrow \int_0^L f(x) dx = \int_0^L x + c_2 dx = \frac{L^2}{2} + c_2 L\]

\[\Rightarrow c_2 = -\frac{L}{2} + \frac{1}{L} \int_0^L f(x) dx\]

\[\Rightarrow u = x - \frac{L}{2} + \frac{1}{L} \int_0^L f(x) dx\].
Part (c)  \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta \), \( u(x,0) = f(x) \), \( \frac{\partial u}{\partial x}(0,t) = 0 \), \( \frac{\partial u}{\partial x}(L,t) = 0 \)

\( \frac{\partial^2 u}{\partial x^2} = \beta - x \Rightarrow \frac{\partial u}{\partial x} = \beta x - \frac{x^2}{2} + c_1 \Rightarrow u = \frac{\beta x^2}{2} - \frac{x^3}{6} + c_1 x + c_2 \)

0 = \frac{\partial u}{\partial x}(0) = c_1 \Rightarrow c_1 = 0

0 = \frac{\partial u}{\partial x}(L) = \beta L - \frac{L^2}{2} + c_1 = \beta L - \frac{L^2}{2} \Rightarrow \beta = \frac{L}{2}

Only if \( \beta = \frac{L}{2} \), we can find equilibrium solutions.

To determine \( c_2 \), we need to check the total energy is constant in time

\[ \frac{d}{dt} \int_0^L u \, dx = \int_0^L \frac{\partial u}{\partial t} \, dx = \int_0^L \frac{\partial^2 u}{\partial x^2} + x - \beta \, dx \]

\[ = \left[ \frac{\partial u}{\partial x} \right]_0^L - \frac{\partial u}{\partial x}(0,1) + \frac{x^2}{2} \bigg|_0^L - \beta x \bigg|_0^L \]

\[ = 0 - 0 + \frac{L^2}{2} - \beta L = \frac{L^2}{2} - \frac{L}{2} L = 0 \quad \checkmark \]

\[ \Rightarrow \int_0^L f(x) \, dx = \int_0^L u(x) \, dx = \int_0^L \left( \frac{\beta x^2}{2} - \frac{x^3}{6} + c_2 \right) \, dx \]

\[ = \frac{L}{4} x^3 \bigg|_0^L - \frac{x^4}{24} \bigg|_0^L + c_2 L = \frac{L}{4} \frac{L^3}{3} - \frac{L^4}{24} + c_2 L \]

\[ = \frac{L^4}{24} + c_2 L \Rightarrow c_2 = -\frac{L^4}{24} + \frac{1}{L} \int_0^L f(x) \, dx \]

\[ \Rightarrow u = \frac{L}{4} x^2 - \frac{x^3}{6} - \frac{L^3}{24} + \frac{1}{L} \int_0^L f(x) \, dx \]