\[
\begin{align*}
\frac{1}{2c} \int_{x-c}^{x+c} G(x) \, dx &= \frac{1}{2c} \sum_{n=1}^{\infty} B_n \frac{n \pi c}{L} \sin \frac{n \pi x}{L} \, dx \\
&= \frac{1}{2c} \sum_{n=1}^{\infty} B_n \frac{n \pi c}{L} \left( \cos \frac{n \pi x}{L} \right)_{x-c}^{x+c} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} B_n \cos \left( \frac{n \pi c}{L} (x+c) \right) - B_n \cos \left( \frac{n \pi c}{L} (x-c) \right) \\
&= \sum_{n=1}^{\infty} B_n \sin \frac{n \pi c}{L} \sin \frac{n \pi x}{L} = u(x, t)
\end{align*}
\]

Therefore, \( u(x, t) = \frac{1}{2c} \int_{x-c}^{x+c} G(x) \, dx \).

**Problem 3:** From (4.4.1) derive the conservation of energy for a vibrating string,

\[
\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x} \left| \frac{\partial u}{\partial t} \right|_0
\]

where the total energy \( E \) is the sum of the kinetic energy, defined by \( \int_0^L \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 \, dx \), and the potential energy, defined by \( \int_0^L \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 \, dx \).

**Answer:**

\[
E(t) = \int_0^L \left( \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \frac{c^2}{2} \left( \frac{\partial u}{\partial x} \right)^2 \right) \, dx
\]

\[
\frac{dE}{dt} = \int_0^L \frac{\partial}{\partial t} \left( \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \frac{c^2}{2} \left( \frac{\partial u}{\partial x} \right)^2 \right) \, dx
\]
\[
\frac{dE(t)}{dt} = \int_0^l \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \frac{\partial^2 u}{\partial x^2} \right) \, dx
\]

Because \( u \) satisfies the wave eqn.

\[
= c^2 \int_0^l \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} \, dx
\]

If you don't know what to do, integrate by parts:

\[
c^2 \int_0^l \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} \, dx = \left. \frac{c^2}{2} \frac{\partial u}{\partial t} \right|_0^l - c^2 \int_0^l \frac{\partial^2 u}{\partial x^2} \, dx
\]

\[
= \left. \frac{c^2}{2} \frac{\partial u}{\partial t} \right|_0^l - c^2 \int_0^l \frac{\partial^2 u}{\partial x^2} \, dx
\]

\[
\Rightarrow \frac{dE}{dt} = \frac{c^2}{2} \frac{\partial u}{\partial t} \bigg|_0^l - c^2 \int_0^l \frac{\partial^2 u}{\partial x^2} \, dx
\]

Problem 4: Let \( u(x, t) = \sum_{n=1}^{\infty} B_n \cos \left( \frac{n \pi c t}{L} \right) \sin \left( \frac{n \pi x}{L} \right) \)

be a solution of the vibrating string problem. Suppose the string is constrained so that \( u(x, 0) = 0 \) for all \( t \). What conditions does this impose on the coefficients \( B_n \)?

\( O = u(L/3, t) = \sum_{n=1}^{\infty} B_n \cos \left( \frac{n \pi c t}{L} \right) \sin \left( \frac{n \pi x}{3} \right) \)

\( 0 = u(L/3, t) = \sum_{n=1}^{\infty} B_n \cos \left( \frac{n \pi c t}{L} \right) \sin \left( \frac{n \pi x}{3} \right) \)

\( \Rightarrow \) Since \( \cos \left( \frac{n \pi c t}{L} \right) \) form a linearly independent set of functions, then

\( B_n \sin \left( \frac{n \pi x}{3} \right) = 0 \)

\( \sin \left( \frac{n \pi x}{3} \right) = 0 \) only when \( \frac{n}{3} \) is an integer
\( B_n = 0 \) if \( n \) is not a multiple of 3.

That is, \( B_n = 0 \) if \( n = 3k+1 \) or \( 3k+2 \), \( k \) integer.

**Problem 5:** The voltage \( V(x,t) \) in a transmission cable is known to satisfy the PDE:

\[
\frac{\partial^2 V}{\partial t^2} + 2a \frac{\partial V}{\partial t} + a^2 V = c^2 \frac{\partial^2 V}{\partial x^2},
\]

where \( a \) and \( c \) are positive constants. Let \( u(x,t) = e^u(x,t) \) and show that \( u \) satisfies the wave equation \( U_{tt} = c^2 U_{xx} \).

\[
U_t = ae^{at} V + e^{at} \frac{\partial V}{\partial t}
\]

\[
U_{tt} = a e^{at} V + ae^{at} \frac{\partial V}{\partial t} + ae^{at} \frac{\partial^2 V}{\partial t^2} + e^{at} \frac{\partial^2 V}{\partial t^2}
\]

\[
= ae^{at} V + 2ae^{at} \frac{\partial V}{\partial t} + e^{at} \frac{\partial^2 V}{\partial t^2}
\]

\[
U_{xx} = e^{at} V_{xx}
\]

\[
\Rightarrow U_{tt} - c^2 U_{xx} = e^{at} \left( a^2 V + 2a \frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial t^2} - c^2 V_{xx} \right)
\]

\[
= e^{at} \cdot 0 = 0
\]

\[
\Rightarrow U_{tt} = c^2 U_{xx}.
\]