ground, which puts it ahead of the off-diagonal sum. For Fig. 2.1, with six edges and three ungrounded nodes,

\[ A^TCA = \begin{bmatrix} c_1 + c_2 + c_5 & -c_1 & -c_2 \\ -c_1 & c_1 + c_3 + c_4 & -c_3 \\ -c_2 & -c_3 & c_2 + c_3 + c_6 \end{bmatrix}. \] (7)

The matrices \( A^T A \) and \( A^T CA \) are symmetric positive definite "M-matrices," mentioned in an earlier footnote (Section 1.4). Apart from zeros, they are negative off the diagonal and all entries in their inverses are positive.

4. Every area of applied mathematics has its own interpretation of \( A, C, b, \) and \( f. \) Some problems also have their own choice of sign; each section will translate between the specific application and the common framework. In certain cases (for example structures) the sign of \( x \) is reversed, and so is the sign of \( A: \)

\[ \begin{bmatrix} C^{-1} & -A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix}. \] (8)

It is now \( + A^T CA \) which appears halfway through elimination, and therefore all the pivots will be positive. But the matrix in (8) is no longer symmetric.

5. Strictly speaking Fig. 2.1 represents a directed graph. It is a network when a number \( c_i \) is assigned to every edge.

6. It is remarkable how well the framework extends to problems that are continuous rather than discrete. Instead of a finite number of edges, the flow may fill a region in the plane. The unknowns change from vectors to functions, and the potential differences change to derivatives; they still produce the flow. It is governed by differential equations instead of matrix equations. Nevertheless the pattern in Fig. 2.2 still leads to \( A^T CA, \) and describes the equilibrium of the system—this is the theme of Chapter 3.

**EXERCISES**

2.1.1 For a graph with edges around a square and across one diagonal \( (N = 4 \) and \( m = 5) \), number the nodes and edges and write down the incidence matrix \( A_0. \) What is \( A_0^T A_0? \)

The next three exercises refer to the network with four nodes and six edges at the beginning of the section. The edge constants are \( c_1, \ldots, c_6. \)

2.1.2 (a) Compute the 4 by 4 matrices \( A_0^T A_0 \) and \( A_0^T CA_0 \) for the network in Fig. 2.1. Notice that like the original \( A_0, \) its columns add up to the zero column.
(b) Verify that removing the last row and column of $A_0^T C A_0$ leaves $A^T C A$ in equation (7). What is $A^T A$?

(c) Show that this $A^T A$ is positive definite by applying one of the tests in Chapter 1 (for example, compute the determinants or the pivots).

2.1.3 For the triangular network in Fig. 2.1, let $f_1 = f_2 = f_3 = 1$ and $f_4 = -3$. With $C = I$ and $b = 0$, solve the equilibrium equation $-A^T C A x = f$. (Note that $f_4$ and $x_4$ do not enter, because $x_4 = 0$ and the last column of $A_0$ was removed.) Solve also for $y$, and describe the flows through the network.

2.1.4 Suppose there are "batteries" $b_4 = b_5 = b_6 = 1$ on the inner edges of the network in Fig. 2.1. With $f = b_1 + b_2 + b_3 = 0$ and $C = I$, write down the 9 by 9 equilibrium system (4). Show that there is no flow: the solution has $y = 0$ (but not $x = 0$).

2.1.5 For a network with only $m = 3$ edges and $N = 3$ nodes, at the vertices of a triangle with arrows clockwise, write down $A_0$ and $A$. With $f = 0$, $b_1 = b_2 = b_3 = 1$, and $C = I$, find $x$ and $y$.

2.1.6 Suppose a network has $N$ nodes and every pair is connected by an edge. Find $m$, the number of edges.

2.1.7 Imagine an $R$ by $R$ network in the plane, with nodes at the $N = R^2$ points with integer coordinates between 1 and $R$.

(a) If horizontal and vertical edges make it into a network of unit squares find the number of edges. Show that the approximate ratio of $m$ to $N$ is 2 to 1.

(b) If the network also includes the diagonals of slope 1 in each square, adding $(R - 1)^2$ new edges, show that the approximate ratio of $m$ to $N$ for this triangular mesh is 3 to 1.

2.1.8 Show that the particular matrix

$$M = \begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is neither positive definite nor negative definite, by finding its pivots (and also its eigenvalues). Does

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1x_2$$

have a minimum or maximum or saddle point at $x_1 = x_2 = 0$?

2.1.9 Given a network and its incidence matrices (ungrounded and grounded):

(i) find $A^T C A$ from equations (5) and (6), with $-c_1$, $-c_2$, $-c_3$ appearing off the diagonal
(ii) find $A^TCA$ from "column-row" multiplications, where the first column of $A^T$ and the first row of $A$ give
\[
\begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
c_1 \\
-c_1 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
c_1 & -c_1 & 0 \\
-c_1 & c_1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Add up the five products of this kind (one for each edge).

2.1.10 Eliminate $y$ to find the equations for $x$ in the systems
\[
\begin{bmatrix}
C^{-1} & A \\
A^T & D
\end{bmatrix}
\begin{bmatrix}
y \\
x
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
f
\end{bmatrix}
\quad \text{and} \quad 
\begin{bmatrix}
C_1^{-1} & 0 & A_1 \\
0 & C_2^{-1} & A_2 \\
A_1^T & A_2^T & 0
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
x
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
f
\end{bmatrix}.
\]

2.1.11 What is the equation for the vector $x$ that minimizes $P(x) = \frac{1}{2}(b - Ax)^T C (b - Ax)$?

2.1.12 Draw a network with no loops (a tree). Check that with one node grounded the incidence matrix $A$ is square, and find $A^{-1}$. All entries of the inverse are 1, $-1$, or 0.

2.1.13 Suppose $A_0$ is a 12 by 9 incidence matrix from a connected graph. Its exact form is unknown but it has 12 edges and 9 nodes, none grounded.

(a) How many columns of $A_0$ are independent?

(b) How many rows are independent, and what do the corresponding edges look like on the graph?

(c) What condition on $f$ makes $A_0^T y = f$ solvable?

(d) How many independent solutions are there to $A_0^T y = 0$, and how can they be found from the graph? (See page 74.)

(e) For which vectors $b$ does $A_0 x = b$ have a solution?