That removes irrelevant edges. A large $\sigma$ leaves only the big picture; a small $\sigma$ allows a closer look. Pattern recognition is an inverse problem—to recover the coloring book from the finished picture—like recovering the coefficients of a differential equation from its solutions.

**EXERCISES**

3.3.1 (a) Show that $u = x^3 - 3xy^2$ satisfies Laplace's equation.

(b) Do the same for $s = 4x^3y - 4xy^3$, and explain where this comes in the list of polynomial solutions.

(c) Substitute $x = \cos \theta$ and $y = \sin \theta$ in $s$ and simplify to an expression involving $4\theta$.

3.3.2 Verify that $u = e^x \cos y$ and $s = e^x \sin y$ both satisfy Laplace's equation, and sketch the equipotentials $u =$ constant and the streamlines $s =$ constant.

3.3.3 *Discrete divergence theorem:* Why is the flow across the “cut” in the figure equal to the sum of the flows from the individual nodes $A,B,C,D$? *Note:* This is true even if flows like $d_1 - d_6$ from nodes like $A$ are nonzero. If the current law holds and each node has zero net flow, then the exercise says that the flow across every cut is zero.

3.3.4 *Discrete Stokes theorem:* Why is the voltage drop around the large triangle equal to the sum of the drops around the small triangles? *Note:* This is true even if voltage drops like $d_1 + d_2 + d_6$ around triangles like $ABC$ are nonzero. If the voltage law holds and the drop around each small triangle is zero, then the exercise says that $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0$.

3.3.5 On a graph the analogue of the gradient is the edge-node incidence matrix $A_0$. The analogue of the curl is the loop-edge matrix $R$ with a row for each independent loop and a column for each edge. Draw a graph with four nodes and six directed edges, write down $A_0$ and $R$, and confirm that $RA_0 = 0$ in analogy with curl grad $= 0$.

3.3.6 Why does the flow rate $w = (\partial s/\partial y, -\partial s/\partial x)$ satisfy $\text{div } w = 0$ for any “stream function” $s(x,y)$?
3.3.7 If the density is \( c = 1 \) then

\[
\begin{bmatrix}
\frac{\partial s}{\partial y} \\
\frac{-\partial s}{\partial x}
\end{bmatrix}
\]
is equal to \( v = \begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{bmatrix} \).

Show from these Cauchy-Riemann equations \( \frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x} \) that both \( u \) and \( s \) satisfy Laplace’s equation.

3.3.8 The curves \( u(x,y) = \text{constant} \) are orthogonal to the family \( s(x,y) = \text{constant} \) if grad \( u \) is perpendicular to grad \( s \). These gradient vectors are at right angles to the curves, which can be equipotentials and streamlines. Construct a suitable \( s(x,y) \) from the geometry and verify

\[
(\text{grad } u)^T (\text{grad } s) = \frac{\partial u}{\partial x} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial s}{\partial y} = 0
\]

(a) \( u(x,y) = y \): equipotentials are parallel horizontal lines

(b) \( u(x,y) = x - y \): equipotentials are parallel 45° lines

(c) \( u(x,y) = \log(x^2 + y^2)^{1/2} \): equipotentials are concentric circles.

3.3.9 A differential equation like \( dy/dx = f(x,y) \) gives a family of curves depending on the initial value \( y(0) \), and \( dy/dx = -1/f(x,y) \) gives the orthogonal curves. (The product of the slopes is \(-1\), the usual condition for a right angle; the gradients are in the orthogonal directions \((1,f)\) and \((1,-1/f)\).) Solve \( y' = -1/f \) for the second family if the first family is

(a) \( y = e^x + \text{constant} \), from \( dy/dx = e^x = f \)

(b) \( y = \frac{1}{2}x^2 + \text{constant} \), from \( dy/dx = x = f \)

(c) \( xy = \text{constant} \), from \( dy/dx = -y/x = f \).

3.3.10 In Stokes’ law (8), let \( v_1 = -y \) and \( v_2 = 0 \) to show that the area of \( S \) equals the line integral \(-\int_C y \, dx\). Find the area of an ellipse \((x = a \cos t, y = b \sin t, x^2/a^2 + y^2/b^2 = 1, 0 \leq t \leq 2\pi)\).

3.3.11 By computing curl \( v \), show that \( v = (y^2, x^2) \) is not the gradient of any function \( u \) but that \( v = (y^2, 2xy) \) is such a gradient—and find \( u \).

3.3.12 By computing div \( w \), show that \( w = (x^2, y^2) \) does not have the form \((\partial s/\partial y, -\partial s/\partial x)\) for any function \( s \). Show that \( w = (y^2, x^2) \) does have that form, and find the “stream function” \( s \).

3.3.13 If \( u = x^2 \) in the square \( S = \{-1 < x, y < 1\} \), verify the divergence theorem (11) when \( w = \text{grad } u \):

\[
\int \int_S \text{div } \text{grad } u \, dx \, dy = \int_C n \cdot \text{grad } u \, ds.
\]

If a different \( u \) satisfies Laplace’s equation in \( S \), what is the net flow through \( C \)?

3.3.14 What potential has the gradient \( v = (u_x, u_y) = (2xy, x^2 - y^2) \)? Sketch the equipotentials and streamlines for flow into a 30° wedge (Fig. 3.7 was 45°), and show that \( v \cdot n = 0 \) on the upper boundary \( y = x/\sqrt{3} \). The streamlines have \( s = xy^2 - \frac{1}{3}x^3 = \text{constant} \).
3.3.15 Solve Poisson's equation \( u_{xx} + u_{yy} = 4 \) by trial and error if \( u = 0 \) on the circle \( x^2 + y^2 = 1 \).

3.3.16 Find a quadratic solution to Laplace's equation if \( u = 0 \) on the axes \( x = 0 \) and \( y = 0 \) and \( u = 3 \) on the curve \( xy = 1 \).

3.3.17 Laplace's equation in polar coordinates is

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]

Show that \( u = r \cos \theta + r^{-1} \cos \theta \) is a solution, and express it in terms of \( x \) and \( y \). Find \( v = (u_x, u_y) \) and verify that \( v \cdot n = 0 \) on the circle \( x^2 + y^2 = 1 \). This is the velocity of flow past a circle.

3.3.18 Show that \( u = \log r \) satisfies Laplace's equation except at \( r = 0 \).

3.3.19 Suppose \( \delta P/\delta u = \int \int \left[ \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} - fv \right] \, dx \, dy \). Use Green's formula, changing the \( u \) in (17) to \( v \) and changing \( w \) to grad \( u \), to write

\[
\frac{\delta P}{\delta u} = \int_S v \left[ ? \right] \, dx \, dy + \int_C v \left[ ? ? \right] \, ds.
\]

If this is zero for all \( v \), find the differential equation and the natural boundary condition satisfied by \( u \).