Lecture 5: Fundamental equations for equilibrium

The framework discussed last time for graphs is useful in many physical systems:

- $x_j$: potential at node $j$ (N)
- $y_i$: flow on edge $i$ (m)

The $y_i$ are determined by the potential difference across the edges and by physical properties of edges. Physical properties usually represented by an $m \times m$ matrix $C$, often diagonal $(c_1, \ldots, c_m)$.

In any case $C$ is symmetric and $y$ connected to $x$ by $A_0$ $(m \times N)$.

$A_0$ gives the geometry of the network.

Use $+1/-1$ edge convention from last lecture.
For network above,

\[ A_0 = \begin{pmatrix}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 1 \\
-1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1
\end{pmatrix} \]

The columns of \( A_0 \) are not linearly independent: cannot get unique solution to \( Ax = b \).

This is the arbitrary potential at each node discussed in last lecture.

To eliminate this ambiguity, ground one node: \( x_N = 0 \).

Then

\[ A = \begin{pmatrix}
-1 & 1 & 0 \\
-1 & 0 & 1 \\
0 & -1 & 1 \\
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix} \]

(drop last column in \( A_0 \))

is \( m \) by \( n = N - 1 \)

\( Ax = b \) has unique solution.

For mechanical networks \( n \) is the total number of degrees of freedom at the nodes, not counting any nodes that are fixed.

(In other words, eliminate fixed nodes for which \( x \) is known.)
The vectors \( b \) and \( f \) cause things to happen.

\( b \) gives voltage sources.

Then: \( e = b - Ax \) will lead to flow.

**Ohm's law or Hooke's law** is then

\[
y = Ce = C(b - Ax)
\]

Finally, the second fundamental equation is

\[
A^T y = f \quad \text{Kirchhoff's current law}
\]

\( b = 0 \) for Hooke's law

\( f = 0 \)

Note that springs have \( b = 0 \), while circuits have \( f = 0 \).

These can be grouped into the fundamental equations for equilibrium:

\[
\begin{pmatrix}
C^{-1} & A \\
A^T & 0
\end{pmatrix}
\begin{pmatrix}
y \\
x
\end{pmatrix}
=
\begin{pmatrix}
b \\
f
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
C^{-1} & A \\
0 & -A^T C A
\end{pmatrix}
\begin{pmatrix}
y \\
x
\end{pmatrix}
=
\begin{pmatrix}
b \\
f - A^T C b
\end{pmatrix}
\]
So we get $m$ positive pivots and $n$ negative pivots.

For a network,

$$(A^TCA)_{jk} = -c_j \text{ if edge } i \text{ connects } j \neq k$$

$$(A^TCA)_{kk} = \sum c_j \text{ over edges meeting node } k.$$ 

For our earlier example: $A^TCA = \begin{pmatrix} c_1 + c_2 + c_3 & -c_1 & -c_2 \\ -c_1 & c_1 + c_4 & -c_5 \\ -c_2 & -c_3 & c_2 + c_3 + c_5 \end{pmatrix}$
Lagrange multipliers:

Example: minimize $Q = \frac{1}{2} (y_1^2 + y_2^2)$, subject to the constraint $2y_1 - y_2 = 5$.

Solve this by defining $L = Q + \lambda_1 (2y_1 - y_2 - 5)$.

(Note $L = Q$ when constraint is satisfied)

$\partial L/\partial y_1 = y_1 + 2\lambda_1 = 0$ \hspace{1cm} minimize $L(y_1, y_2, \lambda_1)$

$\partial L/\partial y_2 = y_2 - \lambda_1 = 0$

$\partial L/\partial \lambda_1 = 0 + (2y_1 - y_2 - 5) = 0$ \hspace{1cm} constraint satisfied

Solution is $(y_1, y_2, \lambda_1) = (2, -1, -1)$. 

[Diagram showing the intersection of a paraboloid and a line, with a point marked (2, -1).]
Now minimize
\[ Q = \frac{1}{2} y^T C y - b^T y \]

with constraint \( A^T y = f \).

Here \( A \) is \( m \times n \), so \( m \) constraints.

Need Lagrange multipliers \( x_1, \ldots, x_m \)

\[ L = Q + \sum x^T (A^T y - b) \]

\[ \frac{\partial L}{\partial y_i} = \frac{\partial Q}{\partial y_i} + \sum x_j A_{ij} = 0 \quad x^T A^T y = \sum x_j A_{ij} y_j \]

\[ \Rightarrow C y - b + A x = 0 \quad \frac{\partial L}{\partial x_i} = \sum x_j A_{ij} \]

\[ 2(1) = \sum x_j A_{ij} = (A x). \]

Together with the constraint itself, this is

\[
\begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix}
\]

There are the fundamental equations for equilibrium!