Lecture 6: Duality and trusses

Classic example: minimize $Q = y^T Ky$ subject to $y^T y = 1$.

$L = Q + x (y^T y - 1)$

$\frac{\partial L}{\partial y} = 2 (Ky + xy) = 0 \Rightarrow Ky = -xy$

So $x$ is an eigenvalue of $K$.

Since $K$ symmetric, eigenvalues are real, so to minimize take smallest eigenvalue, $x = -1$, with $e$-vec $y$.

Then $\min L = y^T Ky = \lambda y^T y = 1$.

$max L$ gives the matrix 2-norm of $K$.

Equivalently, $\min/\max$ ratio $y^T Ky / y^T y$.

Duality. Primal problem:

Minimize $Q(y) = \frac{1}{2} y^T C^{-1} y - b^T y$ subject to $A^T y = f$.

$\rightarrow L = Q + x^T (A^T y - f)$

$L$ is multi-vector.
\[ \frac{\partial L}{\partial y} = C^{-1} y - b + Ax = 0 \]
\[ \Rightarrow y = C (b - Ax) \]
\[ \min L(x, y) = -\frac{1}{2} (b - Ax)^T C (b - Ax) - x^T f \]
\[ =: -P(x) \quad P(x) \text{ is an "energy"} \]

**Dual problem:**

\[ \text{Maximize} \quad -P(x) = -\frac{1}{2} (Ax - b)^T C (Ax - b) - x^T f \]
\[ \text{(no constraint!)} \]

**Now we show that Dual = Primal.**

**First,** for any \( x \) and \( y \) such that \( A^T y = f \), we have
\[ Q(y) \geq -P(x), \quad "\text{weak dual}" \]
\[ Q(y) \geq L(x, y) \geq -P(x) \]  
\[ \text{"trick" with} \]

Why? Because \( -P \) is the minimum of \( L \)!

So \( L(x, y) \geq -P(x) \), and \( L = Q \) at the min.

**Now take** \( Q + P \):  \( \text{use} \ A^T y = f \)

\[ Q + P = \frac{1}{2} y^T C^{-1} y - b^T y + \frac{1}{2} (Ax - b)^T C (Ax - b) + x^T f \]
\[ = \frac{1}{2} (C^{-1} y + Ax - b)^T C (C^{-1} y + Ax - b) \]
\[ \geq 0 \]
Here, $Q + P = \frac{1}{2} z^T C z$ has a min. at $z = 0$.

$\Rightarrow C^{-1} y + A x - b = 0.$

We reach "full duality" when $z = 0$ ($Q = -P$).

Summary: constrained min. $\{ Q(y) \}$ subject to $A^T y = f$

equals the max $-P(x)$. The minimizing $y$ and maximizing $x$ (saddle pt. $A L$, where $Q + V = 0$)

satisfy:

$C^{-1} y + A x = b$ \hspace{1cm} A^T y = f$

Return to spring example: (y = internal force)

$Q = \frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + \frac{1}{2} y_3^2 = \frac{1}{2} y^T C^{-1} y$

with constraint (force below)

$y_1 - y_2 = f_1 \hspace{1cm} y_2 - y_3 = f_2 \hspace{1cm} (A^T y = f)$

Minimize $Q$ subject to constraint:

$L = Q - x_1 (y_1 - y_2 - f_1) - x_2 (y_2 - y_3 - f_2)$

Minimizing gives $y = C A x \Rightarrow A^T C A x = f$
Could also minimize \( P = \frac{1}{2} x^T A^T C A x - x^T f \), total potential energy

**Structures in equilibrium:**

- **Trusses:** system of idealized bars that doesn’t bend.

Consider 2D configurations.

- Give external forces \( f_j \) at each node
- Two displacements \( x_j \) for each node

Each support fixes 2 displacements, total of \( r \) for supports.

So remains \( n = 2N - r \) degrees of freedom.

\( x_1, \ldots, x_n \) \( \leftrightarrow \) includes horizontal and vertical displacements.

Similarly, there are \( 2N \) components of forces at each node.
We do not know the forces at the supports, but the other $2N - r$ forces are prescribed.

$f_1, \ldots, f_n$ applied forces.

Forces $y_1, \ldots, y_m$ in the bars.

$y_i > 0$ is in tension. No direction needed in the graph.

$y_i < 0$ is in compression.

Recall displacements $e = b - Ax$.

Convention in mechanics is $e = Ax$.

So a positive force is in the direction of positive displacement.

$A$ has $m$ rows, one for every bar.

Suppose the end of a bar of original length $L$ moves.

$(L \cos \theta, L \sin \theta) \rightarrow (L \cos \theta + x_1, L \sin \theta + x_2)$

$l = L + e$
\[ l^2 = (L \cos \theta + x_1 - x_3)^2 + (L \sin \theta + x_2 - x_4)^2 \]

To leading order in \( \kappa \):

\[ l = L + 2L (x_1 \cos \theta - x_3 \cos \theta + x_2 \sin \theta - x_4 \sin \theta) + O(\kappa^2) \]

\[ l = L + \left( \frac{\kappa}{L} \right) + O(\kappa^2/L) \]

\[ e = l - L = \text{elongation} \]

We thus have a typical term in \( A \kappa \).

Nonzero entries of each row are \( \pm \cos \theta, \pm \sin \theta \).

\[ \left[ \begin{array}{c} \Gamma \new 3D, \pm \cos \theta, \pm \cos \phi, \pm \sin \phi, \text{direction cosines} \end{array} \right] \]

Not the sum of each row is still 0.

Now drop the r columns corresponding to each fixed displacement.

Get \( A \), m by n.

\[ A^T y = f \] will then give the force balance.