Lecture 11: Principle of least action: Analytical methods

In dynamics, least energy is replaced by least action.

For an elastic bar,

\[ A(t) = \int_{t_0}^{t_1} (K - P) \, dt \]

where:
- \( K \) is kinetic energy
- \( P \) is potential energy
- \( t \) is time
- \( t_0 \) and \( t_1 \) are initial and final times

\[ = \int_{t_0}^{t_1} \int_0^1 dx \left( \frac{1}{2} \rho \left( \frac{du}{dt} \right)^2 - \frac{1}{2} c \left( \frac{du}{dx} \right)^2 \right) \]

\( \rho \) is density

Can use the same technique as before: \( u + u + u \)

then enforce \( A(t) \) being a minimum (extremum condition)

\[ \frac{\partial}{\partial t} \left( \rho \frac{du}{dt} \right) = \frac{\partial}{\partial x} \left( c \frac{du}{dx} \right) \]

with appropriate BC.

For \( \rho = c = \text{const.} \), this is \( u_{tt} = \frac{\xi}{\rho} u_{xx} \)

the wave equation

\[ A^T C A u = f \]

So here:
- \( A = \left( \begin{array}{c} -\frac{\xi}{\rho} t \\ \frac{\xi}{\rho} x \end{array} \right) \)
- \( A^T = \left( \begin{array}{cc} \frac{\xi}{\rho} t \\ -\frac{\xi}{\rho} x \end{array} \right) \)
- \( C = \left( \begin{array}{cc} \rho & \xi \\ -\xi & \rho \end{array} \right) \)
Analytical methods:

Examples: solve $\nabla^2 u = 0$, $(x, y) \in$ unit disk

$u(1, \theta) = u_0(\theta)$

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0
\]

Separation of variables: $u = R(r) \Theta(\theta)$

\[
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0
\]

$O \quad R'' + \Theta \frac{r'}{r} + \Theta \frac{R'}{r} = 0$

\[
\frac{r^2}{R} \left( R'' + \Theta \frac{r'}{r} \right) = -\Theta'' = \text{const} = \lambda^2
\]

$\Theta$ equation:

$\Theta'' + \lambda^2 \Theta = 0$

$\Theta = \sin \lambda \theta, \cos \lambda \theta$

$\sin \theta$ periodic, $\lambda = m \in \mathbb{Z}$

[Different for wedge problem: $\Delta$.]

$R$ equation: $R \sim r^p$, $\lambda'' + \frac{\lambda'}{r} - \frac{\lambda}{r^2} r^2 = 0$

$p(p-1) + p - \lambda^2 = 0 \iff \lambda^2 = 1^2, \quad \lambda = \pm 1$
Hence, the solutions are $R \sim r^3 \exp(-r)$.

When $\lambda = 0$ there is only one solution: $R \sim \text{const}$.

The other solution: $\frac{1}{r^2} \frac{d}{dr} (r^2 R') = 0 \Rightarrow R' = c_1 \Rightarrow R = c_1 \ln r + c_2$.

So finally:

$$u(r,0) = \sum_{m=0}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta)$$

$$= \sum_{n=-\infty}^{\infty} c_n r^n e^{im\theta}$$

but again blows up at $r=0$.

$$u(1,\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = u_0(\theta)$$

$$\sum_{m=-\infty}^{\infty} c_m \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} u_0(\theta) e^{-i\theta} d\theta$$

The $2\pi f_{m,n}$'s are orthogonal functions.

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} u_0(\theta) e^{-i\theta} d\theta$$

This is a pair solution.
Can also write in terms of integral solution:

\[ u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u_0(\psi)}{1 + r^2 - 2r \cos(\theta - \psi)} \, d\psi \]

Why? Because

\[ \frac{1 - r^2}{2(1 + r^2 - 2r \cos \theta)} = \frac{1}{2} + r \cos \theta + r^2 \cos 2\theta + \cdots \]

So after changing \( \cos(\theta - \psi) = \cos \theta \cos \psi + \sin \theta \sin \psi \),

we see each term is a coefficient of the Fourier series.

So, for example, if \( u_0(\psi) = f(\psi) \),

\[ u(r, \theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos \theta} \]

\( u(r, \theta) \) can be regarded as steady temperature distribution due to a point source of heat, or electric field, ...

Note that the orthogonal functions \( e^{in\theta} \) arose naturally from the equation.