Lecture 18: Chaos part 1: maps

**Poincaré-Bendixson Theorem:** \( \dot{x} = f(x, y), \dot{y} = g(x, y) \)

D closed bounded region of the x-y plane.

\( f, g \) continuously differentiable in D.

If a trajectory remains in D for all \( t \geq 0 \), then as \( t \to \infty \) the trajectory must

1) be a fixed point or closed orbit or

2) approach a fixed point or closed orbit.

See link on website for a nice discussion/proof.

This is not true in other types of region: for example, consider the torus: \( T = C[0,1]^2 \), periodic

\[
\begin{align*}
\dot{x} &= 1 \\
\dot{y} &= x
\end{align*}
\]

For \( x \) irrational, on orbit will never close.

(But we will not call this "chaotic"
So what do we need for chaos?

(We assume motion in a bounded region.)

Need:

- non-autonomous system ("2+1")
- ≥ 3 dimension
- A mapping

Mappings are the easiest way to get chaos:

\[ x_{n+1} = f(x_n) \quad \text{Can be chaotic even in 1D.} \]

Example: Logistic map

\[ x_{n+1} = \alpha \cdot x_n \cdot (1-x_n) \]
In 1D maps, fixed points are at the intersection of the line $y=x$ and the curve $y=f(x)$. 

\[ f(x_*) = x_* \] fixed point

**Stability of fixed point:** $x = x_* + \varepsilon$

\[ x_* + \varepsilon_{n+1} = f(x_* + \varepsilon_n) \]

\[ = f(x_*) + f'(x_*) \varepsilon_n + O(\varepsilon_n^2) \]

\[ \Rightarrow \varepsilon_{n+1} = f'(x_*) \varepsilon_n \]

**Stable if** $|f'(x_*)| < 1$; **monotonic**

**Unstable if** $|f'(x_*)| > 1$.

For logistic map, two fixed points: $x_0 = 0$ and

\[ x_* = \alpha \cdot x_*(1-x_*) \Rightarrow x_* = 1 - \alpha^{-1} \]

For $\alpha > 1$: **monotonic** approach to fixed point

For $\alpha < 1$: **oscillatory** approach to fixed point
\[ f'(x) = \frac{d}{dx} [\alpha x (1-x)] = \alpha (1 - 2x) \]

\[ f'(0) = \alpha \quad \text{stable for } |\alpha| < 1 \]

\[ f'(1-\alpha^{-1}) = \alpha (1 - 2(1-\alpha^{-1})) = 2 - \alpha \quad \text{stable for } |2 - \alpha| < 1 \]

For \( 0 < \alpha < 1 \), \( x_0 = 0 \) is stable, but \( x_0 = 1 - \alpha^{-1} \) unstable.

For \( 1 < \alpha < 3 \), \( x_0 = 0 \) is unstable, but \( x_0 = 1 - \alpha^{-1} \) is stable.

For \( \alpha > 3 \) both are unstable.

Now we can get periodic orbits: \( x_{n+2} = x_n \) (period-2)

\[ x_{n+2} = f(x_{n+1}) = \alpha x_{n+1} (1 - x_{n+1}) = \alpha x_n (1-x_n) (1-\alpha x_n (1-x_n)) \]

Solve \( x_n \) for period-2 orbit:

\[ x = \alpha x (1-x) (1 - \alpha x (1-x)) \]

\( x = 0 \) is still a solution, as is \( x = 1 - \alpha^{-1} \)

(These fixed points)
Two more solutions:

\[ x_{\pm}=\frac{1}{2}(1+\alpha^{-1} \pm \sqrt{(1+\alpha^{-1})(1-3\alpha^{-1})}) \]

These are the two points of the period-2 orbit!

This orbit won’t exist unless \( 1-3\alpha^{-1} > 0 \Rightarrow \alpha > 3 \)

But that coincides with both fixed points losing stability.

This scenario repeats itself as we increase \( \alpha \):

period 2 becomes unstable \( \rightarrow \) period 4 becomes stable, etc.

"period-doubling bifurcations"
But the surprising thing is that the bifurcations occur an infinite number of times in a finite interval of \( \alpha \).

"period-doubling cascade"

\[ \rightarrow \text{CHAOS} \]

\( \alpha = 4 \) is particularly interesting.

Consider a "density" of points \( \rho(x) \geq 0 \), \( x \in [0,1) \).

How is a density mapped forward?

\[ \rho_{n+1}(x) = \int_0^1 \rho_n(y) \delta(x - f(y)) dy \]

\( x = f(y) \) \text{ has two solutions: } \( y_\pm = \frac{1}{2} \left( 1 \pm \sqrt{1-x} \right) \)

\[ f'(y_\pm) = 4(1-2y_\pm) = 4 \left( 1 - 2 \frac{1}{2} \left( 1 \pm \sqrt{1-x} \right) \right) \]

\[ = 4 \left( 1 \mp \sqrt{1-x} \right) = \mp 4 \sqrt{1-x} \]

\[ \rho_{n+1}(x) = \frac{\rho_n(y_+)}{|f'(y_+)|} + \frac{\rho_n(y_-)}{|f'(y_-)|} \delta(x - f(y)) = \sum \delta(y-y_\pm) \]

Note that \( 1 - y_- = 1 - \frac{1}{2} \left( 1 - \sqrt{1-x} \right) = y_+ \)
Assume $p_n(1-x) = p_n(x)$ (symmetry)

$p_n(y_-) = p_n(1-y_+) = p_n(y_+)$

Hence,

$$p_{n+1}(x) = 2 \frac{p_n(y_+(x))}{|f'(y_+(x))|} = \frac{p_n(y_+(x))}{2 \sqrt{1-x}}$$

Can we find the "invariant distribution" $p(x)$?

$$p(x) = \frac{p\left(\frac{1}{2}(1+\sqrt{1-x})\right)}{2 \sqrt{1-x}}$$

Check: $p(x) = \frac{1}{\pi} \frac{1}{\sqrt{\pi (1-x)}}$ satisfies this.

$$p\left(\frac{1}{2}(1+\sqrt{1-x})\right) = \frac{1}{\pi} \left(\frac{1}{2}(1+\sqrt{1-x})\right)^{\frac{1}{2}} \left(1-(1-x)\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\pi} \left(1-(1-x)\right)^{-\frac{1}{2}} = \frac{2}{\pi} \frac{1}{\sqrt{x}}$$

$$= 2 \sqrt{1-x} p(x)$$

What does this mean? This invariant distribution tells us how a cloud of points becomes distributed on $[0,1]$.

[Diagram showing accumulation near edge]
For $x = y$, the map itself also has an exact solution:

$$x_{n+1} = 4x_n(1-x_n)$$

$$\Rightarrow x_n = \sin^2(2^n \theta(x_0))$$

where $x_0 = \sin^2(\theta(x_0))$, so $\theta(x_0) = \sin^{-1}(x_0)$.

Consider two "nearby" trajectories:

starting at $\theta(x_0)$, $\theta(x_0 + \delta)$

$$x^n(x_0 + \delta) - x^n(x_0) = 2^n \sin(2^n \theta(x_0)) \frac{\sin(2^n \theta(x_0)) \delta}{\sqrt{x_0(1-x_0)}} + O(\delta^2)$$

Define:

$$\lambda = \lim_{n \to \infty} \lim_{\delta \to 0} \frac{1}{n} \log \left| \frac{x^n(x_0 + \delta) - x^n(x_0)}{\delta} \right|$$

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \log |x^n(x_0)|$$

\(\lambda\) is the derivative as $\delta \to 0$

"asymptotic rate of separation" Lyapunov exponent

In this case $\lambda = \log 2$

$\lambda > 0$ defines chaos
Npts = 20000; Nit = 50; nbins = 50; r = 4;
x = rand(1,Npts); % random initial points
f = @(r,x) r.*x.*(1-x); % logistic map

figure(1)

% the initial points are uniformly distributed on the interval [0,1]
subplot(2,1,1)
[P,xb] = hist(x,nbins); P = P./trapz(xb,P);
bar(xb,P,1)
title ('initial distribution')

% now iterate the map on the initial points
for i = 1:Nit, x = f(r,x); end

% the final points obey the invariant distribution
subplot(2,1,2)
[P,xb] = hist(x,nbins); P = P./trapz(xb,P);
bar(xb,P,1)
title (sprintf ('distribution after %d iterations',Nit))
if r == 4
   hold on
   xi = linspace (0,1,300); xi = xi (2:end-1);
   plot(xi,1/ pi *( xi.*(1-xi)).^-.5, 'r',' LineWidth ',2) % analytic answer for r=4
   hold off
end

figure(2)

% two trajectories very close to each other initially
delta = 1e-10; x1 = [.1]; x2 = [x1 + delta];

% iterate the map on both trajectories
for i = 1:Nit, x1 = [x1 f(r,x1(end))]; x2 = [x2 f(r,x2(end))]; end
% plot the separation
semilogy(0:Nit,abs(x1-x2))
hold on
semilogy(0:Nit,delta*2.^(0:Nit),'r') % compare to 2^n separation rate
hold off
axis([0 Nit delta 10])
xlabel('n')
ylabel('distance between trajectories')

figure(3)

Nit = 2000; Nplot = 1000; r = linspace(0,4,400); x = .1*ones(size(r));

% iterate the map on the same initial condition, but different r
for i = 1:Nit, x = [x; f(r,x(end,:))]; end

clf, hold on
% plot the last Nplot iterates for each r
for i = 1:Nplot
    plot(r,x(Nit-i-1,:),'.','MarkerSize',2)
end
hold off
xlabel('r')
title('bifurcation diagram for logistic map')
Figure 1: Invariant distribution for the logistic map with $r = 4$.

Figure 2: Separation of nearby trajectories for the logistic map with $r = 4$. The red line is $2^n$. 

Figure 3: Bifurcation diagram for the logistic map. For each $r$, the map is iterated 2000 times starting from $x = 0.1$, and the next 1000 iterates are plotted.