Stokes’ Theorem, Divergence Theorem

**Problem 1.**  A. Use a surface integral to calculate the flux of the curl of the vector field

\[ \nabla \times (x, y, z) = (x^2 y, 2y^3 z, 3z) \]

across a surface patch

\[ S : \mathbf{x}(r, \theta) = (r \cos \theta, r \sin \theta, r) \]

where 0 ≤ r ≤ 1 and 0 ≤ θ ≤ 2π.

B. Use a surface integral to calculate the flux of the curl of the vector field

\[ \nabla \times (y, z, x) = (y^2, z^2, x) \]

across a surface patch

\[ S : \mathbf{x}(\phi, \theta) = (2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi) \]

where 0 ≤ φ ≤ π/2 and 0 ≤ θ ≤ 2π.

**Problem 2.** Use a surface integral to show that the surface area of sphere with radius R is 4πR².

**Problem 3.** Let \( S = \{(x, y, z) : x^2 + y^2 + z^2 = 9, z \geq 0\} \) and \( C \) be the boundary of \( S \) in the direction counter-clockwise with respect to the surface’s outward unit normal vector \( \mathbf{N} \). Let \( \mathbf{F} = y \mathbf{i} - x \mathbf{j} \).

A. By taking a parametrization of \( C \) respecting the above orientation, calculate a line integral

\[ \int_C \mathbf{F} \cdot d\mathbf{x} \]

B. By taking a suitable coordinate patch of \( S \) respecting outward unit normal vector \( \mathbf{N} \), calculate a surface integral

\[ \iint_S (\text{curl} \mathbf{F}) \cdot \mathbf{N} dA \]

C. Verify Stokes’ Theorem in this example.

(Hint: The answer is \(-18\pi\))

**Problem 4.** Use Stokes’ Theorem to evaluate

\[ \int_C \mathbf{F} \cdot d\mathbf{x} \]

if

\[ \mathbf{F} = (xz, xy, 3xz) \]

and \( C \) is the boundary of the portion of the plane \( 2x + y + z = 2 \) in the first octant with counter-clockwise orientation (1, 0, 0) → (0, 2, 0) → (0, 0, 2) → (1, 0, 0). (Hint: The answer is \(-1\))
Problem 5. Let \( \vec{F} = (x^2 - y, 4z, x^2) \). Let \( S \) be the cone given by
\[
\{(x, y, z) : z \leq 2, z = \sqrt{x^2 + y^2}\}.
\]
Let \( C \) be the curve in which the plane \( z = 2 \) meets the cone \( z = \sqrt{x^2 + y^2} \) with the counter-clockwise orientation around the positive \( z \)-axis. Let \( -C \) be the curve in which the plane \( z = 2 \) meets the cone \( z = \sqrt{x^2 + y^2} \) with the clockwise orientation around the positive \( z \)-axis.

A. Take a surface patch to calculate a flux integral of the curl of the vector field \( \vec{F} \).

B. Use Stokes’ theorem to calculate
\[
\iint_C \vec{F} \cdot d\vec{r}, \quad \iint_{-C} \vec{F} \cdot d\vec{r}
\]
(Hint: The answer is \( 4\pi \) or \( -4\pi \))

Problem 6. Let \( \vec{v} = (y - x, z - y, y - x) \). Let \( D \) be the cube bounded by the planes \( x = \pm 1, y = \pm 1, z = \pm 1 \) and \( S \) be the boundary surface. Let \( \vec{N} \) be the outward unit normal on \( S \). Using Divergence Theorem, calculate
\[
\iiint_D \vec{v} \cdot \vec{N} \, dV
\]

Problem 7. Let \( \vec{F} = (x^2, xz, 3z) \). Let \( D \) be the solid sphere given by \( x^2 + y^2 + z^2 \leq 4 \) and \( S \) be the boundary surface. Let \( \vec{N} \) be the outward unit normal on \( S \). Using Divergence Theorem, calculate
\[
\iiint_S \vec{F} \cdot \vec{N} \, dA
\]

Problem 8. Let \( \vec{v} = (x^2, -2xy, 3xz) \). Let \( D \) be the region cut from the first octant by the sphere given by \( x^2 + y^2 + z^2 = 4 \) and \( S \) be the boundary surface. Let \( \vec{N} \) be the outward unit normal on \( S \). Using Divergence Theorem, calculate
\[
\iiint_S \vec{v} \cdot \vec{N} \, dA
\]

Problem 9. Let \( \vec{v} = (6x^2 + 2xy, 2y + x^2z, 4x^2y^3) \). Let \( D \) be the region cut from the first octant by the cylinder given by \( x^2 + y^2 = 4 \) and the plane \( z = 3 \). Let \( S \) be the boundary surface. Let \( \vec{N} \) be the outward unit normal on \( S \). Using Divergence Theorem, calculate
\[
\iiint_S \vec{v} \cdot \vec{N} \, dA
\]

Problem 10. Let \( \vec{F} = (x/\sqrt{x^2 + y^2 + z^2}, y/\sqrt{x^2 + y^2 + z^2}, z/\sqrt{x^2 + y^2 + z^2}) \). Let \( D \) be the region given by \( 1 \leq x^2 + y^2 + z^2 \leq 4 \). Let \( \vec{N} \) be the outward unit normal on \( S \). Using Divergence Theorem, calculate
\[
\iiint_S \vec{F} \cdot \vec{N} \, dA
\]

Reference: These problems are taken from Thomas’ Calculus 11th edition.