1. As another approach to developing a compact method for producing the $LU$ factorization of $A$, consider the following matrix-oriented approach. Write

$$A = \begin{pmatrix} \hat{A} & d \\ c^T & \alpha \end{pmatrix}, \quad c, d \in \mathbb{R}^{n-1}, \quad \alpha \in \mathbb{R}$$

and $\hat{A}$ square of order $n - 1$. Assume $A$ is nonsingular. As a step in an induction process, assume $\hat{A} = \hat{L}\hat{U}$ is known, with $\hat{A}$ nonsingular. Look for $A = LU$ in the form

$$A = \begin{pmatrix} \hat{L} & 0 \\ m^T & 1 \end{pmatrix} \begin{pmatrix} \hat{U} & q \\ 0 & \gamma \end{pmatrix}, \quad m, q \in \mathbb{R}^{n-1}, \quad \gamma \in \mathbb{R}$$

Show that $m, q$ and $\gamma$ can be found, and describe how to do so. (This method is applied to an original $A$, factoring each principle submatrix in the upper left corner, in increasing order.)

2. Let $A$ and $B$ have order $n$, with $A$ nonsingular. Consider solving the linear system

$$Az_1 + Bz_2 = b_1, \quad Bz_1 + Az_2 = b_2$$

with $z_1, z_2, b_1, b_2 \in \mathbb{R}^n$.

(a) Find necessary and sufficient conditions for convergence of the iteration method

$$Az_1^{(m+1)} = -Bz_2^{(m)} + b_1, \quad Az_2^{(m+1)} = -Bz_1^{(m)} + b_2, \quad m \geq 0$$

(b) Repeat part (a) for the iteration method

$$Az_1^{(m+1)} = -Bz_2^{(m)} + b_1, \quad Az_2^{(m+1)} = -Bz_1^{(m+1)} + b_2, \quad m \geq 0$$

Compare the convergence rates of the two methods.

3. Let $C_0$ be an approximate inverse to $A$. Define $R_0 = I - AC_0$, and assume $\|R_0\| < 1$ for some matrix norm. Define the iteration method

$$C_{m+1} = C_m(I + R_m) \quad R_{m+1} = I - AC_{m+1}, \quad m \geq 0$$

This is a well-known iteration method for calculating the inverse $A^{-1}$. Show the convergence of $C_m$ to $A^{-1}$ by first relating the error $A^{-1} - C_m$ to the residual $R_m$. And then examine the behavior of the residual $R_m$ by showing $R_{m+1} = R_m^2, \quad m \geq 0$. 

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4. The system $Ax = b,$

$$A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 \\
-1 & 0 & 0 & 4 & -1 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\
1 \\
1 \\
2 \\
2 \\
2 \end{pmatrix}$$

has the solution $x = [1, 1, 1, 1, 1]^T.$ Solve the system using the Jacobi iteration method, and then solve it again using the Gauss-Seidel method. Use the initial guess $x^{(0)} = 0.$ Note the rate at which the iteration error decreases. Find the answers with an accuracy of $\epsilon = .0001.$