1. Let the function \( f : \mathbb{R} \to \mathbb{R} \) have the zero \( \xi \). Let \( f \) be twice continuously differentiable and satisfy \( f'(x) \neq 0 \) for all \( x \in I := \{ x : |x - \xi| \leq r \} \). The method

\[
x_{n+1} := x_n + \frac{f(x_n)^2}{f(x_n) - f(x_n + f(x_n))}, \quad n = 0, 1, \ldots,
\]

is a quasi-Newton method. Show:

(a) The method has the form

\[
x_{n+1} := x_n - q(x_n)f(x_n).
\]

Give \( q(x_n) \), and show that there is a constant \( c \) such that

\[
|q(x) - f'(x)^{-1}| \leq c |f(x)|.
\]

(b) One can construct a majorizing sequence \( y_n \) sufficiently close to \( \xi \),

\[
|x_n - \xi| \leq y_n \quad \text{for all} \quad n \geq 0.
\]

Give conditions which ensure that \( y_n \) converges to zero. Using \( y_n \), determine the local order of convergence.

(c) Test this method numerically for some functions that you are curious about that have zeros. Compute the error defined as \( |x_{n+1} - x_n|/|x_n - x_{n-1}| \) and observe the convergence rate.

2. Suppose \( A_0 \in \mathbb{R}^{n \times n} \) is symmetric and positive definite and consider the following iteration:

\[
\begin{align*}
&\text{for} \quad k = 1, 2, \ldots \\
&A_{k-1} = G_k G_k^T \quad \text{(Cholesky)} \\
&A_k = G_k^T G_k
\end{align*}
\]

(a) Show that this iteration produces matrices that have the same eigenvalues as \( A_0 \)

(b) Show that if

\[
A_0 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}
\]

with \( a \geq c \) has eigenvalues \( \lambda_1 \geq \lambda_2 > 0 \), then the \( A_k \) converges to \( \text{diag}(\lambda_1, \lambda_2) \).

3. Apply the QR method to the following matrix

\[
A = \begin{pmatrix}
-1.6407 & 1.0814 & 1.2014 & 1.1539 \\
1.0814 & 4.1573 & 7.4035 & -1.0463 \\
1.2014 & 7.4035 & 2.7890 & -1.5737 \\
1.1539 & -1.0463 & -1.5737 & 8.6944
\end{pmatrix}
\]

Take 10 iterations and observe how the matrix is being converging to an upper triangular matrix. Print out all matrices. You can implement this algorithm easily if you use MATLAB.