1. Consider the singular value decomposition of a real, nonsingular \( n \times n \) matrix \( A \),
\[
U^T A V = D
\]
where \( U \) and \( V \) are real orthogonal matrices, 
\[
D = \text{dial}(\mu_1, \cdots, \mu_n)
\]
with \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0 \).

Take the Euclidean norm as the underlying vector norm. Let \( x \) be the solution to \( Ax = b \),
and the solution \( x + \Delta x \) corresponds to the right-hand side \( b + \Delta b \),
\[
A(x + \Delta x) = b + \Delta b.
\]

(a) Express \( \text{cond}(A) \) in terms of the singular values \( \mu_i \).

(b) Prove the following bounds:
\[
\begin{align*}
\| \Delta x \| &\leq \| A^{-1} \| \| \Delta b \|, \\
\frac{\| \Delta x \|}{\| x \|} &\leq \text{cond}(A) \frac{\| \Delta b \|}{\| b \|}.
\end{align*}
\]

(c) Give an expression for the vectors \( b \) and \( \Delta b \) in terms of \( U \) for which the bounds in (b) and
\[
\| b \| \leq \| A \| \| x \|
\]
are satisfied with equality.

(d) Is there a vector \( b \) such that for all \( \Delta b \) in (1.2),
\[
\frac{\| \Delta x \|}{\| x \|} \leq \frac{\| \Delta b \|}{\| b \|}
\]
holds? Determine such vectors \( b \) with the help of \( U \).

2. Consider the 2-by-2 matrix
\[
A = \begin{pmatrix}
1 & \rho \\
-\rho & 1
\end{pmatrix}.
\]

(a) Under what conditions will Gauss-Seidel converge with this matrix?

(b) For what range of \( \omega \) will the SOR method converge? What is the optimal choice for this parameter?

(c) Repeat (a) and (b) for the matrix
\[
A = \begin{pmatrix}
I_n & S \\
-S^T & I_n
\end{pmatrix},
\]
where \( S \in \mathbb{R}^{n \times n} \). (Hint: Use the SVD of \( S \)).

3. Show that if \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite and has \( k \) distinct eigenvalues,
then the conjugate gradient method does not require more than \( k + 1 \) steps to converge.