content

1. vector and rules of calculation (dot product, cross product, etc.)

2. parametric curves (e.g., velocity, arc-length, unit tangent vector, curvature, normal and binormal vector torsion, etc.)

3. multi-variable function (e.g., function of plane, quadratic forms, polar coordinate, graph and level set)

4. partial derivative (e.g., linear approximation, tangent plane, implicit function theorem, coordinate transformation, Clairaut’s theorem etc.)

reviewing material

1. Previous exams;

2. Homework questions;

3. lecture notes and book;

4. quiz;

1. vector

A vector is an array of numbers: \( \vec{a} = (x, y, z) \). \( \vec{a} \) is determined by its length and direction.

\[
|\vec{a}| = \sqrt{x^2 + y^2 + z^2}
\]

Also remember, the unit vector in the same direction of \( \vec{a} \) can be found by: \( \frac{\vec{a}}{|\vec{a}|} \).

Arithmetic rules of vectors:

\[
(x, y, z) + (a, b, c) = (x + a, y + b, z + c)
\]

\[
c(x, y, z) = (cx, cy, cz)
\]

Dot product: (the result is a number)

\[
(x, y, z) \cdot (a, b, c) = xa + yb + zc
\]

cross product: (the result is a vector)

\[
(x, y, z) \times (a, b, c) = (yc - zb, za - xc, xb - ya)
\]

\[
\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta; \ \theta \text{ is the angle between } \vec{a}, \vec{b}
\]

\[
|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta; \ \theta \text{ is the angle between } \vec{a}, \vec{b}
\]

Thumb rules:

1. \( \vec{a} \) is perpendicular to \( \vec{b} \) is equivalent to say \( \vec{a} \cdot \vec{b} = 0 \);

2. \( \vec{a} \times \vec{b} \) is perpendicular to \( \vec{a}, \vec{b} \); (it follows right hand rule) (so if you know the plane \( P \) is expanded by \( \vec{a}, \vec{b} \), then you can find the normal vector to the plane by: \( \vec{a} \times \vec{b} \).
3. \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \), \( \vec{a} \times \vec{a} = 0 \);

4. The volume of the parallelepiped generated by vector \( \vec{a}, \vec{b}, \vec{c} \) is: \( Vol = |\vec{a} \cdot (\vec{b} \times \vec{c})| \);

5. If you know \( \vec{a} \) is a point in the plane \( P \), \( \vec{n} \) is the normal vector to the plane, then the function of \( P \) is given by: \( (\vec{x} - \vec{a}) \cdot \vec{n} = 0 \);

exercise:

1. \( \vec{a} = (1, 2, 3), \vec{b} = (-1, 0, 1) \), calculate \( |\vec{a}|, \vec{a} \cdot \vec{b}, \vec{a} \times \vec{b} \), the unit vector in the same direction of \( \vec{a} \).

2. Chapter 1, 12. question: 5, 8, 12, 17

3. Show that, if \( \vec{a} \) is vertical to \( \vec{b} \), then \( |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2, |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \).

4. \( P \) is a plane which intersects with \( x, y, z \) axises at points: \((1, 0, 0), (0, 1, 0), (0, 0, 1) \). Determine the function of \( P \).

5. Quiz 4, question 2(a)

2. parametric curves

parametrized curve: \( \vec{x}(t) = (x(t), y(t), z(t)) \).

examples:
a line: \( \vec{x}(t) = \vec{a} + t\vec{v} \);
circle: \( \vec{x}(t) = (R\cos(\omega t), R\sin(\omega t)) \)
cycloid: \( \vec{x}(t) = (Rt - Rsint, R - Rcost) \)
helix: \( \vec{x}(t) = (R\cos(t), R\sin(t), t) \)

arithmetic rules:

the derivative: (velocity) \( \vec{x}'(t) = (x'(t), y'(t), z'(t)) \)
second derivative: (acceleration) \( \vec{x}''(t) = (x''(t), y''(t), z''(t)) \)
\[
(\vec{a} \cdot \vec{b})' = \vec{a}' \cdot \vec{b} + \vec{a} \cdot \vec{b}'
\]
\[
(\vec{a} \times \vec{b})' = \vec{a}' \times \vec{b} + \vec{a} \times \vec{b}'
\]
unit tangent vector: \( \hat{T} = \frac{\vec{x}'}{|\vec{x}'|} \)
curvature vector: \( \hat{\kappa} = \frac{\hat{T}''}{|\vec{x}'|} \)
curvature: \( \kappa = |\hat{\kappa}| \)
normal vector: \( \hat{N} = \frac{\hat{\kappa}}{\kappa} \)
bi-normal vector: \( \vec{B} = \vec{T} \times \vec{N} \)

torsion: \( \tau = \frac{|\vec{B}'|}{|\vec{x}'|} \)

arclength: \( l = \int_a^b |\vec{x}'(t)|dt \)

eample: helix \( \vec{x}(t) = (cost, sint, t) \).

\[
\vec{x}'(t) = (-sint, cost, 1)
\]

\[
|\vec{x}'(t)| = |(-sint, cost, 1)| = \sqrt{(-sint)^2 + (cost)^2 + 1^2} = \sqrt{2}
\]

\[
\vec{x}''(t) = (-cost, -sint, 0)
\]

\[
\vec{T}(t) = \frac{\vec{x}'(t)}{|\vec{x}'(t)|} = \frac{1}{\sqrt{2}}(-sint, cost, 1)
\]

\[
\vec{\kappa}(t) = \frac{\vec{T}'(t)}{|\vec{x}'(t)|} = \frac{1}{\sqrt{2}}(-cost, -sint, 0)
\]

\[
\kappa = |\vec{\kappa}| = \frac{1}{2}
\]

\[
\vec{N}(t) = \frac{\vec{\kappa}}{\kappa} = (-cost, -sint, 0)
\]

\[
\vec{B}(t) = \vec{T} \times \vec{N} = \frac{1}{\sqrt{2}}(-sint, cost, 1) \times (-cost, -sint, 0) = \frac{1}{\sqrt{2}}(sint, -cost, 1)
\]

\[
\tau = \frac{|\vec{B}'(t)|}{|\vec{x}'(t)|} = \frac{1}{2}|(cost, sint, 0)| = \frac{1}{2}
\]

the arc-length from time 0 to time 2\( \pi \) is: \( l = \int_0^{2\pi} |\vec{x}'(t)|dt = \int_0^{2\pi} \sqrt{2}dt = 2\sqrt{2}\pi \)

exercises:

1. Chapter 2. 17, question 1, 4, 7

(remark for 7: if the parametrized curve is restricted in a plane: \( \vec{x}(t) = (x(t), y(t), 0) \). Then \( \vec{B}(t) = (0, 0, 1) \) or \( (0, 0, -1) \), and \( \tau = 0 \))

3. multi-variable function

some concepts:

the graph of a function \( z = f(x, y) \) is: \( \{(x, y, z) : z = f(x, y)\} \).

the c-level set of a function \( z = f(x, y) \) is: \( f^{-1}(c) = \{(x, y) : f(x, y) = c\} \). The most common one is 0-level set: \( f^{-1}(0) = \{(x, y) : f(x, y) = 0\} \). for example, if \( f(x, y) = x^2 + y^2 \), then \( f^{-1}(-1) = \emptyset, f^{-1}(0) = \{(0,0)\}, f^{-1}(1) = \{(x, y) : x^2 + y^2 = 1\} \) which is the unit circle.

quadratic forms: \( f(x, y) = Ax^2 + Bxy + Cy^2 \).
some examples:
positive definite form: \( f(x, y) = x^2 + y^2 \); negative definite form: \( f(x, y) = -x^2 - y^2 \);
semi-definite form: \( f(x, y) = x^2, f(x, y) = -y^2 \);
indeterminate form: \( f(x, y) = xy, f(x, y) = x^2 - y^2 \).

Classify quadratic forms:
\( f(x, y) = Ax^2 + 2Bxy + Cy^2 \). Assume \( A \neq 0 \). Then by completing the square: \( f(x, y) = A[(x + \frac{B}{2A}y)^2 + \frac{4AC-B^2}{4A^2}y^2] \)
\( \Delta = 4AC - B^2 \) is called the discriminant. If \( \Delta > 0 \), \( f(x, y) \) is definite; if \( \Delta = 0 \), \( f(x, y) \) is semi-definite; if \( \Delta < 0 \), \( f(x, y) \) is indeterminate.

example: \( f(x, y) = x^2 + 6xy + 4y^2 = (x + 3y)^2 - 5y^2 \), \( f(x, y) \) is indeterminate.

polar coordinate:
\( x = r \cos \theta, y = r \sin \theta \), where \( r \) denotes the radius, \( \theta \) denotes the polar angle.
If \( \theta = \arctan \left( \frac{y}{x} \right) \), then the range of \( \theta \) is \((-\frac{\pi}{2}, \frac{\pi}{2})\), the valid domain is: \( \{(x, y) : x > 0\} \).

examples of graphs of multi-variable functions:
check \( z = \theta, z = \sin \theta \) on textbook page 44, 45.

Exercises
1. Chapter 3 page 46, question 3, 5, 8, 10, 13.

4. partial derivative

when you compute the partial derivative of \( f(x, y) \) over variable \( x \), consider the variable \( y \) as a constant and do the usual derivative over \( x \). The same thing goes for \( y \). For example, \( f(x, y) = \ln(x^2y + 3xy^3), \partial_x f = \frac{1}{x^2y + 3xy^3} \cdot (2xy + 3y^3) = \frac{2x+y^3}{x^2y + 3xy^3}, \partial_y f = \frac{1}{x^2y + 3xy^3} \cdot (x^2 + 9xy^2) = \frac{x+9y^2}{xy + 3y^3} \).

linear approximation:
\[
f(x, y) \approx f(x_0, y_0) + \partial_x f(x_0, y_0) \cdot (x - x_0) + \partial_y f(x_0, y_0) \cdot (y - y_0)
\]

By the formula of linear approximation, we can find the tangent plane:
\[
z = z_0 + \partial_x f(x_0, y_0) \cdot (x - x_0) + \partial_y f(x_0, y_0) \cdot (y - y_0)
\]

example:
\( f(x, y) = x^2 + y^2 \), \( (x_0, y_0) = (1, 1) \).
Then: \( f(x_0, y_0) = 2, \partial_x f(1, 1) = 2, \partial_y f(1, 1) = 2 \). The tangent plane at \((1, 1, 2)\) is:
\[
z = 2 + 2(x - 1) + 2(y - 1)
\]
which is: \( z = 2x + 2y - 2 \)
gradient: \( \nabla f(x, y, z) = (\partial_x f(x, y, z), \partial_y f(x, y, z), \partial_z f(x, y, z)) \). Intuitively, it can be thought as the changing ratio of function value. We can also use gradient to rewrite the linear approximation formula:

\[
f(x, y, z) \approx f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)
\]

So if \((x_0, y_0, z_0) \in \) the level set \( f^{-1}(c) \), then we can find the tangent plane of \( f^{-1}(c) \) at \((x_0, y_0, z_0)\) by:

\[
\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0
\]

chain-rule for multi-variable function:

\[
\frac{df}{dt}(x(t), y(t)) = \partial_x f(x(t), y(t)) \cdot x'(t) + \partial_y f(x(t), y(t)) \cdot y'(t)
\]

example 1:
Let \( f(x, y) = x^2 + y^2 - 1 \), It’s easy to check \((0, 1)\) is a point of \( f^{-1}(0) \). Find the tangent line at \((0, 1)\).

We know at the level set \( f^{-1}(0) \), the value of function \( f \) is constant. Then \( \frac{df}{dx} = 0 \), \( \nabla f \) is vertical to the tangent line. \( \Rightarrow \):

\[
\partial_x f(0, 1)(x - 0) + \partial_y f(0, 1) \cdot (y - 1) = 0
\]

\( \Rightarrow y = 1 \)

example 2:
Let \( f(x, y) = x^2 + y^2 + z^2 \). Then \((1, 1, 1)\) is a point at the level set \( f^{-1}(3) \). Find the tangent plane of \( f^{-1}(3) \) at \((1, 1, 1)\).

\[
\nabla f(1, 1, 1) \cdot (x - 1, y - 1, z - 1) = 0
\]

\( \Rightarrow \)

\[
(2, 2, 2) \cdot (x - 1, y - 1, z - 1) = 0
\]

\( \Rightarrow \)

\[
x + y + z - 3 = 0
\]

check the example of "the tangent at the origin" on page 65.

implicit function theorem:
Consider the differentiable implicit equation \( F(x, y) = 0 \). If at \((x_0, y_0)\), \( F(x_0, y_0) = 0 \) and \( \partial_y F(x_0, y_0) \neq 0 \), then near the point \((x_0, y_0)\), \( y \) can be represented as a function of \( x \): \( y = y(x) \), and \( y'(x) = -\frac{\partial_x F(x, y)}{\partial_y F(x, y)} \).
example:
The unit circle: $x^2 + y^2 = 1$ is a level set of the function: $F(x, y) = x^2 + y^2$. $(0, 1)$ is a point in this level set. Use the implicit function theorem to represent $y$ as a function of $x$ near the point $(0, 1)$.

$\partial_y F(0, 1) = 2 \neq 0$, then we can apply the implicit function theorem to this problem. Near the point $(0, 1)$, $y'(x) = -\frac{\partial_x F(x, y)}{\partial_y F(x, y)} = \frac{-2x}{2y} = -\frac{x}{y}$. Then we have:

$$y'(x) = -\frac{x}{y}$$

separate variables: $y \cdot y'(x) = -x$
which is: $\frac{1}{2}(y^2)' = -x$
integrate both side: $\int_1^y \frac{1}{2}(s^2)' ds = \int_0^x -tdt$

simplify: $y^2 - 1 = -x^2$
Since $y_0 = 1 > 0$, $y$ is near $y_0$, so $y > 0$. and $y = \sqrt{1 - x^2}$.

exercise: Consider the function: $F(x, y, z) = (\sqrt{x^2 + z^2} - 2)^2 + y^2 - 1$. The shape of the 0-level set: $T = F^{-1}(0)$ is the shape of a torus.(or donut)

a On the torus $T$, find out the set of points where $\partial_z F(x, y, z) \neq 0$. (near those points, the 3-dimensional implicit function theorem also works: $z$ can be represented as a function of $x, y$. That’s why we are interested at those points)

b Now consider $z$ as a function of $x, y$. So we have the implicit function for $z$: $(\sqrt{x^2 + z(x, y)^2} - 2)^2 + y^2 = 1$. Find the critical points of $z(x, y)$, i.e., those points where $\partial_x z = 0$ and $\partial_y z = 0$ at the same time. Find the 1 level set of $z(x, y)$: $z^{-1}(1)$. What’s the shape of the level set?

Homework section 10, question 13: Let $f(x, y) = \ln(2 + 2x + e^y)$.

a compute the gradient of $f$ at the point $(x_0, y_0)$ with position vector $\vec{x}_0 = (1, 0)$.

b You are allowed to choose a point at a distance 0.01 from the point $(1, 0)$. Where would you choose the new point if you want $f$ to be as large as possible?

c Is your answer to the previous the exact answer, or only an approximation? i.e., could someone else find a point at distance 0.01 from $(1,0)$ at which $f$ has a (slightly) higher value than at the point you found?

d The level set $C$ of $f$ through the point $(1, 0)$ happens to be the graph of a function $y = g(x)$. Find that function.

e Find normal vector to the tangent line to $C$ at the point $(1,0)$. Find an equation for the tangent line to $C$ at $(1,0)$.
f. How much is \( g(1) \)? Find two different ways to compute \( g'(1) \) based on the work you have done so far.

\[ \nabla f(x, y) = \left( \frac{2}{2 + 2x + e^y}, \frac{e^y}{2 + 2x + e^y} \right). \]
so \( \nabla f(1, 0) = \left( \frac{2}{5}, \frac{1}{5} \right) \).

b. The direction is the direction of the gradient and the magnitude is 0.01: \( \vec{v} = 0.01 \cdot \frac{\nabla f(1, 0)}{|\nabla f(1, 0)|} = \frac{\sqrt{5}}{100} \left( \frac{2}{5}, \frac{1}{5} \right) \).

c. It’s the linear approximation. Since the gradient is not a constant vector at different points.

d. since \( (1, 0) \) is on the level set, \( f(1, 0) = \ln(5) \), \( C \) is the level set of \( \ln(5) \). \( C \) is:
\[ 2 + 2x + e^y = 5 \]
which is:
\[ y = \ln(3 - 2x) \]

e. The gradient at \( (1, 0) \) is vertical to the tangent line. Thus, the function of tangent line at \( (1,0) \) is:
\[ (2, 1) \cdot (x - 1, y) = 0 \]
\[ \Rightarrow y + 2x - 2 = 0 \]

f. \( g(1) = \ln(1) = 0 \).
The first way to find \( g'(1) \) is to do the derivative directly.
The second way to find \( g'(1) \) is by using the implicit function: \( g'(1) = -\frac{\partial_x f(1,0)}{\partial_y f(1,0)} = -\frac{2}{e^y}(1,0) = -2 \).
**Chain rule and coordinate transformation**

The chain rule is an analogue to the one variable case:

One variable case: If $f(u) = F(x(u))$, then $f'(u) = F'(x) \cdot x'(u)$;

Multi-variable case: If $f(u, v) = F(x(u, v), y(u, v))$, then $\partial_u f(u, v) = \partial_x F \cdot \partial_u x + \partial_y F \cdot \partial_u y$.

Example:

By polar coordinate system, $x = r \cos \theta$, $y = r \sin \theta$. Let $f(r, \theta) = F(x(r, \theta), y(r, \theta))$. Then:

$$\partial_r f(r, \theta) = \partial_x F \cdot \cos \theta + \partial_y F \cdot \sin \theta$$

$$\partial_\theta f(r, \theta) = -\partial_x F \cdot r \cdot \sin \theta + \partial_y F \cdot r \cdot \cos \theta$$

Coordinate transformation:

There is a point $p$ in the two dimensional plane. $p$ has coordinate $(x, y)$ under $A$’s axises; $p$ has coordinate $(X, Y)$ under $B$’s axises. $B$ is achieved by rotating $A$ by $\theta$ degree. So the scenario is: $p$ is the fixed point, but it has different coordinates under different systems $A$ and $B$. Then we have the coordinate transformation formula:

$$x = X \cos \theta - Y \sin \theta$$

$$y = X \sin \theta + Y \cos \theta$$

It’s also easy to get the inverse formula:

$$X = x \cos(-\theta) - y \sin(-\theta)$$

$$Y = x \sin(-\theta) + y \cos(-\theta)$$

since rotate $A$ by $\theta$ degree you will get $B$; conversely rotate $B$ by $-\theta$ degree you will get $A$.

Example:

Homework Section 12 question 14:

Brian and Ally are using different Cartesian coordinate systems in the plane: $(x, y)$ for Ally and $(X, Y)$ for Brian. They have the same origin, but Brian’s coordinates are rotated by an angle of $\theta = \arctan(\frac{4}{3})$.

a What is the relation between $(x, y)$ and $(X, Y)$?

b If Ally has found that $T_A(x, y) = 32 + 0.1y$, then what formula $T_B(X, Y)$ will Brian use to describe the temperature?

c On a different occasion Ally found that the temperature had changed. Now Ally measures the temperature and finds that at the point with $x = 1, y = 1$ one has $T_A(1, 1) = 35$ and also $\partial_x T_A = 0.05$ and $\partial_y T_A = 0.8$. Which coordinates does Brian assign to this point? Which temperature $T_B$, and which derivatives $\partial_X T_B$ and $\partial_Y T_B$ does Brian compute at this point?
a.

Let’s initial Ally’s coordinate system by $A$, Brian’s coordinate system by $B$.
Then $B$ is achieved by rotating $A$ by an angle $\theta = \arctan(\frac{4}{3})$.
By the coordinate transformation formula:

$$x = X \cos \theta - Y \sin \theta$$
$$y = X \sin \theta + Y \cos \theta$$

Since $\tan \theta = \frac{4}{3}$, $\sin \theta = \frac{4}{5}$ and $\cos \theta = \frac{3}{5}$.
Then:

$$x = \frac{3}{5}X - \frac{4}{5}Y$$
$$y = \frac{4}{5}X + \frac{3}{5}Y$$

b.

Since $T_A(x, y) = 32 + 0.1y$, then

$$T_B(X, Y) = T_A(x(X, Y), y(X, Y)) = 32 + 0.1y(X, Y) = 32 + 0.1\left(\frac{4}{5}X + \frac{3}{5}Y\right)$$

c.

By the coordinate transformation formula, we have:

$$1 = \frac{3}{5}X - \frac{4}{5}Y$$
$$1 = \frac{4}{5}X + \frac{3}{5}Y$$

Then the coordinate for $X, Y$ is:

$$(X, Y) = (\frac{7}{5}, -\frac{1}{5})$$

$$T_B\left(\frac{7}{5}, -\frac{1}{5}\right) = T_A(1, 1) = 35$$

By chain rule, $T_B(X, Y) = T_A(x(X, Y), y(X, Y))$:

$$\partial_X T_B = \partial_x T_A \cdot \partial_X x + \partial_y T_A \cdot \partial_X y = 0.05 \cdot \frac{3}{5} + 0.8 \cdot \frac{4}{5} = \frac{47}{500}$$
$$\partial_Y T_B = \partial_x T_A \cdot \partial_Y x + \partial_y T_A \cdot \partial_Y y = 0.05 \cdot -\frac{4}{5} + 0.8 \cdot \frac{3}{5} = \frac{1}{125}$$

Clairaut’s theorem:
If for a given function $f$ of two variables the mixed partial derivative $f_{xy}(x, y)$ exists for all $(x, y)$ in a neighborhood of a point $(a, b)$, and if this derivative is continuous at $(a, b)$, then the other mixed partial derivative $f_{yx}(a, b)$ also exists, and $f_{xy}(a, b) = f_{yx}(a, b)$.

The importance of the Clairaut’s theorem is, it’s a necessary condition to check if we can
find a function from its partial derivatives:
The necessary condition for the existence of a function \( f(x, y) \) such that \( \partial_x f(x, y) = P(x, y) \), \( \partial_y f(x, y) = Q(x, y) \) is: \( \partial_y P(x, y) = \partial_x Q(x, y) \).

Example:
1. Let \( P(x, y) = x^3 - 2xy, Q(x, y) = 3y^2 \). Then \( \partial_y P(x, y) = -2x, \partial_x Q(x, y) = 0 \). Since \( \partial_y P(x, y) \neq \partial_x Q(x, y) \), there is no \( f(x, y) \) such that \( \partial_x f = P \) and \( \partial_y f = Q \).

2. Let \( P(x, y) = \cos x + 2x, Q(x, y) = -\sin y + 2y \) defined on a rectangle \( R = \{(x, y) : -1 < x < 1, -1 < y < 1\} \). Then \( \partial_y P(x, y) = 0, \partial_x Q(x, y) = 0 \). Now \( \partial_y P(x, y) = \partial_x Q(x, y) \), the necessary condition is satisfied. The following theorem shows the existence of function \( f \):

Theorem 14.3: Suppose \( P(x, y) \) and \( Q(x, y) \) are two functions that are defined on a rectangular domain \( R = \{(x, y) : a < x < b, c < y < d\} \), and suppose that they have continuous partial derivatives on this domain.

If \( \partial_y P(x, y) = \partial_x Q(x, y) \) in \( R \), then there is a function \( f \) defined on \( R \) such that \( \partial_x f(x, y) = P(x, y) \) and \( \partial_y f(x, y) = Q(x, y) \).

For the example 2 above, by Theorem 14.3, there exists a function \( f(x, y) \) on \( R \) such that \( \partial_x f(x, y) = P(x, y), \partial_y f(x, y) = Q(x, y) \). Now let’s find out the function \( f(x, y) \).

By \( \partial_x f = \cos x + 2x \), integrate both side, we have: \( f(x, y) = \sin x + x^2 + L(y) \). Since \( \partial_y f(x, y) = -\sin y + 2y \), we have: \( L'(y) = -\sin y + 2y \). Then \( L(y) = \cos y + y^2 + C \) where \( C \) is a constant. Thus \( f(x, y) = \sin x + x^2 + \cos y + y^2 + C \). The last step is to check if the function we found really satisfies \( \partial_x f = \cos x + 2x, \partial_y f = -\sin y + 2y \): By direct calculation, \( \partial_x f(x, y) = \cos x + 2x, \partial_y f(x, y) = -\sin y + 2y \). Thus it’s the function we want.

Another example of finding the function \( f \) by \( \partial_x f \) and \( \partial_y f \):
Let \( P(x, y) = 3x^2 + 2xy^2, Q(x, y) = 3x^2y^2 + 2 \) defined on the rectangle \( R = \{(x, y) : -1 < x < 1, -1 < y < 1\} \).

\( \partial_y P(x, y) = 6xy^2 = \partial_x Q(x, y) \). Then by Theorem 14.3, there exists such a function \( f \) on \( R \) such that \( f_x = P, f_y = Q \).

\( f_x = x^2 + 2xy^3 \), integrate both side, we have: \( f(x, y) = x^3 + x^2y^2 + L(y) \).

\( f_y = 3x^2y^2 + 2 \), we have: \( 3x^2y^3 + L'(y) = 3x^2y^2 + 2 \), \( L(y) = 2y + C \).

Then we have got: \( f(x, y) = x^3 + x^2y^2 + 2y + C \), where \( C \) is a constant.

The last step is to check if \( f(x, y) \) is the function we want:
By direct calculation, \( f_x = 3x^2 + 2xy^2 = P, f_y = 3x^2y^2 + 2 = Q \). Thus \( f(x, y) \) is the correct function.