Circle your TA’s name from the following list.

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Please inform your TA if you find any errors in the solutions.

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**Instructions**

- Write neatly on this exam. If you need extra paper, let us know.
- On Problems 1, 2, and 3, only the answer will be graded.
- On Problems 4, 5, 6, and 7 you must show your work and we will grade the work and your justification, and not just the final answer.
- Problem 3 is worth 10 points. All other problems worth 15 points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write \( \cos(\arcsin x) = \sqrt{1 - x^2} \). Note that we have provided some formulas on the next page to help with this.
Formulas

- \( T \infty e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \)
- \( T \infty \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \)
- \( T \infty \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \)
- \( T \infty \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \)
- \( T \infty \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \)
- \( T \infty (1 + x)^b = \sum_{k=0}^{\infty} \binom{b}{k} x^k \) where \( \binom{b}{k} = \frac{b(b-1)(b-2)\cdots(b-k+1)}{k!} \)
1. For each statement below, CIRCLE the correct answer. You do not need to show your work.

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True or false:

(a) \((x \cos(x) - x)\) is \(o(x^5)\).

(b) If \(f(x)\) is a degree 5 polynomial then \(T_{15}f(x) = f(x)\).

(c) \(R_4 \sin x = \sin x - \left(x - \frac{x^3}{3!}\right)\)
Below are three direction fields. The equations for two of those fields are given below. Match the equation to the appropriate direction field and record your answer on the previous page.

(d) \( \frac{dy}{dx} = x^2 - y \)

(e) \( \frac{dy}{dx} = xy \)

Solution:
(a) False.
(b) True.
(c) True.
(d) I.
(e) III.
2. (a) Use Euler’s method with step size $h = 0.1$ to estimate $y(0.1)$ where $y(x)$ satisfies

$$\frac{dy}{dx} = x + y$$

and $y(0) = 1$.

\[ \begin{array}{c|c|c|c}
 x_k & y_k & m_k = x_k + y_k & y_{k+1} = y_k + m_k h \\
\hline
 x_0 = 0 & y_0 = 1 & 0 + 1 = 1 & 1 + (1)(0.1) = 1.1 \\
 x_1 = 0.1 & y_1 = 1.1 & & \\
\end{array} \]

Thus $y(0.1) \approx 1.1$.

(b) Find $T_3 \left(x^3 + \frac{1}{1+2x^2}\right)$.

\textbf{Solution:} $T_{\infty}(x^3 + \frac{1}{1+2x^2}) = T_{\infty}(x^3) + T_{\infty}(\frac{1}{1+2x^2}) = x^3 + (1 - 2x^2 + o(x^3))$. Thus

$$T_3 \left(x^3 + \frac{1}{1+2x^2}\right) = 1 - 2x^2 + x^3$$

(c) Find $T_2^1(x^3)$.

\textbf{Solution:} We compute:

- $f(x) = x^3$
- $f'(x) = 3x^2$
- $f''(x) = 6x$

Thus:

- $f(1) = 1$
- $f'(1) = 3$
- $f''(1) = 6$

This yields:

$$T_2^1(x^3) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2$$

$$= 1 + 3(x - 1) + \frac{6}{2!}(x - 1)^2.$$
3. In the problem below: 1. Clearly define variables (including units!); 2. Set up the appropriate differential equation; and 3. Write down the appropriate initial condition. DO NOT SOLVE THE DIFFERENTIAL EQUATION.

Ten thousand dollars is deposited in a bank account on January 1, 1990 with a nominal annual interest rate of 5% compounded continuously. No further deposits are made. Money is withdrawn continuously at a rate of $4000 per year. We are interested in a function that models the amount of money left in the account.

- **Variables (2pts):**

- **Differential equation (6pts)**

- **Initial condition (2pts):**

**Solution:**

- **Variables (2pts):** $t = \text{time in years since January 1, 1990.} \ M(t) = \text{money in dollars in the bank account at time } t.$

- **Differential equation (6pts)** $\frac{dM}{dt} = .05M - 4000.$

- **Initial condition (2pts):** $M(0) = 10,000.$
4. Find a solution to each initial value problem.

(a) \[ \frac{dy}{dx} = 4x^3(y + e^{x^4}) \quad \text{and} \quad y(0) = 1. \]

Solution:

\[ \frac{dy}{dx} - 4x^3y = 4x^3e^{x^4} \]

This is linear with \( a(x) = -4x^3 \) and \( k(x) = 4x^3e^{x^4} \). Thus

\[ m(x) = e^{\int -4x^3 \, dx} = e^{-x^4}. \]

We check that this satisfies the differential equation \( \frac{dm}{dx} = m(x)a(x) \). We have

\[ \frac{dm}{dx} = e^{-x^4} \cdot (-4x^3) \quad \text{while} \quad m(x)a(x) = e^{-x^4} \cdot (-4x^3). \]

Since these are equal, this passes the sanity check. We thus have:

\[
\begin{align*}
y &= \frac{1}{m(x)} \int m(x)k(x) \, dx \\
&= \frac{1}{e^{-x^4}} \int e^{-x^4} \cdot 4x^3e^{x^4} \, dx \\
&= e^{x^4} \int 4x^3 \, dx \\
&= e^{x^4}(x^4 + C)
\end{align*}
\]

The initial condition \( y(0) = 1 \) gives \( 1 = e^0(0 + C) = C \) and thus \( y = e^{x^4}(x^4 + 1) \).

(b) \[ \frac{1}{\sqrt{1 - y^2}} \frac{dy}{dx} = \cos x \quad \text{and} \quad y(0) = 0. \]

Solution:

\[
\begin{align*}
\frac{dy}{dx} &= \sqrt{1 - y^2} \cos x \\
\frac{dy}{\sqrt{1 - y^2}} &= \cos x \, dx \\
\arcsin(y) &= \sin x + C \\
y &= \sin(\sin x + C)
\end{align*}
\]

Solving initial condition gives \( C = 0 \) and we get \( y = \sin(\sin x) \).
5. Let \( t \) stand for time in minutes from 12:00pm and let \( B(t) \) denote the number of bacteria in a petri dish at time \( t \). Assume that \( B \) satisfies \( \frac{dB}{dt} = 50 \cdot B \cdot (1 - B) \). Also assume that at 12:00pm there were 2 bacteria in the dish. Compute \( B(t) \).

**Solution:** This is a separable differential equation. So we set up and solve:

\[
\int \frac{dB}{B(1-B)} = \int 50 \, dt \quad \text{or} \quad B = 0 \text{ or } B = 1
\]

But \( B = 0 \) and \( B = 1 \) are impossible because they do not satisfy the initial condition. So we can throw them out and move on. Solving the partial fractions problem we get \( \frac{1}{B(1-B)} = \frac{1}{B} + \frac{1}{1-B} \). Thus

\[
\int \frac{1}{B} + \frac{1}{1-B} = \int 50 \, dt
\]

\[
\ln |B| - \ln |1-B| = 50t + C
\]

\[
\ln \left| \frac{B}{1-B} \right| = 50t + C
\]

\[
\frac{B}{1-B} = e^{50t+C} = e^C e^{50t}
\]

\[
\frac{B}{1-B} = \pm e^C e^{50t}
\]

At this point, it might be helpful to use our initial condition to solve for \( C \). Since \( B(0) = 2 \) we get \( \frac{2}{2} = \pm e^C e^0 \) and thus \(-2 = \pm e^C \) and so we choose the negative sign and let \( C = \ln 2 \).

\[
\frac{B}{1-B} = -e^{\ln 2} e^{50t} = -2 e^{50t}
\]

\[
B = (1 - B)(-2 e^{50t}) = -2 e^{50t} + 2Be^{50t}
\]

\[
B = -2Be^{50t}
\]

\[
B = -2e^{50t}
\]

\[
B = \frac{-2e^{50t}}{1-2e^{50t}}
\]

Thus \( B(t) = \frac{-2e^{50t}}{1-2e^{50t}} \).
6. Let \( f(x) = \sin(2x) \). Find \( n \) such that \( |f(x) - T_n f(x)| \leq \frac{1}{100} \) for \( x \) in the range \(-\frac{1}{2} \leq x \leq \frac{1}{2} \).

*It may be helpful to know that* \( 2! = 2, \ 3! = 6, \ 4! = 24, \ 5! = 120 \) and \( 6! = 720 \).

**Solution:** For any \( n \) we have that \( f^{(n+1)}(x) \) is either \( \pm 2^n \sin 2x \) or \( \pm 2^n \cos 2x \). In either case, this function is bounded by \( 2^n \) so we can choose \( M = 2^n \) for any choice of \( n \). Then we have a bound of the following form:

\[
|f(x) - T_n f(x)| \leq \frac{M \cdot c^{n+1}}{(n+1)!} = \frac{2^n \cdot \left(\frac{1}{2}\right)^{n+1}}{(n+1)!} = \frac{1}{(n+1)!}.
\]

So we need to choose \( n \) so that \( \frac{1}{(n+1)!} \leq \frac{1}{100} \). Here we use the fact that \( 5! = 120 \). Thus \( n + 1 = 5 \) and \( n = 4 \) will work.
7. Let \( f(x) \) be a function satisfying the differential equation

\[
f''(x) + 2e^{2x^2} - f(x) = 0\]

and also satisfying the initial conditions \( f(0) = 0 \) and \( f'(0) = -1 \). Compute \( T_4 f(x) \).

**Solution:** We write \( f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4) \). By definition of the Taylor series of \( f(x) \), we have that \( a_0 = f(0) \) and \( a_1 = f'(0) \). Thus \( a_0 = 0 \) and \( a_1 = -1 \). Thus

- \( f(x) = 0 - x + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4) \).
- \( f'(x) = -1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + o(x^3) \).
- \( f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + o(x^2) \).
- \( T_{\infty} 2e^{2x^2} = 2T_{\infty} e^{x^2} = 2(1 + 2x^2 + \frac{(2x^2)^2}{2!}) + o(x^4) = 2 + 4x^2 + 8x^4 + o(x^4) \)

We compute

\[
0 = f''(x) + 2e^{2x^2} - f(x) = (2a_2 + 6a_3 x + 12a_4 x^2 + o(x^2)) + (2 + 4x^2 + 8x^4 + o(x^4)) - (-x + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4))
\]

\[
= (2a_2 + 2 - 0) + (6a_3 + 0 + 1)x + (12a_4 + 4 - a_2) x^2 + o(x^2)
\]

By equating coefficients we see that \( 0 = 2a_2 + 2 \) and thus \( a_2 = -1 \). Also \( 6a_3 + 1 = 0 \) and so \( a_3 = -\frac{1}{6} \). And finally \( 12a_4 + 4 - a_2 = 0 \) but \( a_2 = -1 \) and so \( a_4 = -\frac{5}{12} \). We conclude that

\[
T_4 f(x) = -x - x^2 - \frac{1}{6} x^3 - \frac{5}{12} x^4
\]