Worksheet 4

MATH 222, Week 4: 2.1,2.2,2.3,2.5

Name: ____________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Problem 1. Compute \( \int \frac{x}{\sqrt{x^2-1}} dx \)

Solution 1.

You could solve this using trig sub or rational substitution, but \( u \)-sub turns out to be the best option. If we let \( u = x^2 - 1 \). Then \( du = 2x \, dx \). Substituting we have:

\[
\int \frac{x}{\sqrt{x^2-1}} \, dx = \frac{1}{2} \int \frac{1}{u^{1/2}} \, du = \sqrt{x^2 + 1} + C
\]

Problem 2. Let \( a \) be some constant. Compute the following integral in two ways \( \int \frac{1}{\sqrt{x^2-a^2}} \, dx \).

Solution 2.

The two options are trig sub or rational sub. We’ll do both and compare answers. Option (a) is trig sub, letting \( x = a \sec(\theta) \) or option (b) for rational sub with \( x = aU(t) \).

(a) If we make the trig sub we have:

\[
\int \frac{a \sec(\theta) \tan(\theta)}{a \tan(\theta)} \, d\theta = \int \sec(\theta) \, d\theta = \ln |\sec(\theta) + \tan(\theta)| + C
\]

Now we need to revert back to \( x \) using triangles! We know that \( \sec(\theta) = x/a \) so if we draw our triangle we find that \( \tan(\theta) = \sqrt{x^2-a^2} \). Substituting this back in we have:

\[
\int \frac{x}{\sqrt{x^2-1}} \, dx = \ln \left| \frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a} \right| + C
\]

(b) Using rational substitution we have:

\[
\int \frac{aU'(t)}{aV(t)} \, dt = \int \frac{1 - 1/t^2}{t - 1/t} \, dt = \int \frac{1}{t} \left( \frac{t - 1/t}{t - 1/t} \right) \, dt = \ln |t| + C
\]

We now need to solve for \( t \) in terms of \( x \). We know that \( t = U + V, U = x/a \) and \( V = \sqrt{U^2 - 1} = \sqrt{(x/a)^2 - 1} \). So our final solution is:

\[
\int \frac{x}{\sqrt{x^2-1}} \, dx = \ln \left| \frac{x}{a} + \sqrt{(x/a)^2 - 1} \right| + C
\]

Notice that these are the same answer. What must we do to see they are the same?
Problem 3. Compute \( \int \frac{(z+3)^2}{(40-6z-z^2)^{3/2}} \, dz \)

Solution 3.

We must first complete the square:

\[
40 - 6z - z^2 = -(z + 3)^2 - 7^2
\]

Substituting this back in we have:

\[
\int \frac{(z + 3)^2}{(40 - 6z - z^2)^{3/2}} \, dz = \int \frac{(z + 3)^2}{(49 - (z + 3)^2)^{3/2}} \, dz
\]

Now if we let \( z + 3 = 7 \sin(\theta) \). Then \( dz = 7 \cos(\theta) \, d\theta \). Substituting we have:

\[
\int \frac{(z + 3)^2}{(49 - (z + 3)^2)^{3/2}} \, dz = \int \frac{\sin^2(\theta)}{\cos^2(\theta)} \, d\theta = \int \tan^2(\theta) \, d\theta
\]

We have to be slightly crafty here. We don’t necessarily know how to take the antiderivative of \( \tan^2(\theta) \) but we do for \( \sec^2(\theta) \) and we know \( \tan^2(\theta) = \sec^2(\theta) - 1 \). So we have:

\[
\int \tan^2(\theta) \, d\theta = \int \sec^2(\theta) - 1 \, d\theta = (\tan(\theta) + \theta) + C
\]

Using our handy triangle again we find that \( \tan(\theta) = \frac{z + 3}{\sqrt{49 - (z + 3)^2}} \) and that \( \theta = \arcsin(\frac{z + 3}{7}) \). Substituting this back in we have:

\[
\int \frac{z + 3}{(40 - 6z - z^2)^{3/2}} \, dz = \frac{z + 3}{\sqrt{49 - (z + 3)^2}} + \arcsin \left( \frac{z + 3}{7} \right) + C
\]

\( \square \)

Problem 4. Compute \( \int \frac{e^x}{\sqrt{e^{2x} - 1}} \, dx \). You may find problem 2 helpful.

Solution 4.

Once we make a \( u \)-sub this becomes an exact application of problem 2. Let \( u = e^x \) then \( du = u \, dx \). Substituting we have:

\[
\int \frac{e^x}{\sqrt{e^{2x} - 1}} \, dx = \int \frac{1}{\sqrt{u^2 - 1}} \, du
\]

Now we apply the formula we discovered in problem 2 and substitute back to find:

\[
\int \frac{e^x}{\sqrt{e^{2x} - 1}} \, dx = \ln |\sqrt{e^{2x} - 1} + e^x| + C
\]

\( \square \)

Problem 5. (a) Compute \( \int_0^{\infty} \frac{x}{\sqrt{1+x^2}} \, dx \)

(b) Compute \( \int_{-\infty}^{0} \frac{x}{\sqrt{1+x^2}} \, dx \)

(c) What does this say about \( \int_{-\infty}^{\infty} \frac{x}{\sqrt{1+x^2}} \, dx \)

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Solution 5.

(a) We can use \( u \)-sub here. If we let \( u = 1 + x^2 \) then \( du = 2x \, dx \). Substituting this in we have

\[
\int_0^\infty \frac{x}{\sqrt{1 + x^2}} \, dx = \frac{1}{2} \int_0^\infty \frac{1}{u^{1/2}} \, du
\]

Now we set up our improper integral as a limit:

\[
\frac{1}{2} \int_0^\infty \frac{1}{u^{1/2}} = \lim_{a \to \infty} \frac{1}{2} \int_0^a \frac{1}{u^{1/2}}
\]

Antidifferentiating,

\[
\lim_{a \to \infty} \frac{1}{2} \int_0^a 1 \, du^{1/2} = \lim_{a \to \infty} u^{1/2}\bigg|_0^a = \lim_{a \to \infty} \sqrt{1 + x^2}\bigg|_0^a = \lim_{a \to \infty} \sqrt{1 + a^2} - \sqrt{1} = \infty
\]

(b) We’ll skip right to the limit of the antiderivative here because we already did the work in part (a). Just remember that now we are taking the limit as \( b \to -\infty \) where \( b \neq a \):

\[
\lim_{b \to -\infty} \sqrt{1 + x^2}\bigg|_b^0 = \lim_{b \to -\infty} \sqrt{1 - \sqrt{1 + b^2} = -\infty
\]

(c) We know that

\[
\int_{-\infty}^{\infty} \frac{x}{\sqrt{1 + x^2}} \, dx = \lim_{b \to -\infty} \int_b^0 \frac{x}{\sqrt{x^2 + 1}} \, dx + \lim_{a \to \infty} \int_0^a \frac{x}{\sqrt{x^2 + 1}} \, dx = -\infty + \infty
\]

So the limit diverges.

\[
\square
\]

Problem 6. (a) Show that \( \int_1^\infty \frac{dx}{x^2 - 4} \) is not a finite number.

(b) What answer do you get if you forget to account for the asymptote at \( x = 2 \)?

Solution 6.

(a) To see this we only need to show that one of the pieces of the integral is infinite. Take:

\[
\int_1^2 \frac{dx}{x^2 - 4}
\]

There is an asymptote at \( x = 2 \) and so to compute this improper integral we need to take a limit as \( a \to 2 \) from below:

\[
\int_1^2 \frac{dx}{x^2 - 4} = \lim_{a \to 2} \int_1^a \frac{dx}{x^2 - 4}
\]
Using partial fractions we find the antiderivative:

\[
\lim_{a \to 2} \int_1^a \frac{dx}{x^2 - 4} = \lim_{a \to 2} \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right|_1^a
\]

As \( a \to 2 \) we see that \( \frac{x - 2}{x + 2} \to 0 \) and so the log goes to \(-\infty\). Hence

\[
\lim_{a \to 2} \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right|_1^a = -\infty
\]

So this integral cannot be finite.

(b) If we forget about the asymptote we end up with

\[
\lim_{a \to \infty} \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right|_1^a
\]

In this case as \( a \to \infty \) \( \frac{a - 2}{a + 2} \to 1 \) and so the log goes to 0. Hence

\[
\lim_{a \to \infty} \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right|_1^a = \lim_{a \to \infty} \frac{1}{4} \ln \left| \frac{a - 2}{a + 2} \right|_1 - \frac{1}{4} \ln |1/3| = 0 + \frac{1}{4} \ln(3)
\]

We get a finite number.