Problem 1. Find the general solution to the differential equation:

\[
\frac{dy}{dx} = \frac{1}{e^y \sqrt{1 - x^2}}
\]

Solution 1.

This is separable. If we multiply both sides by \( e^y \) and integrate

\[
\int e^y \, dy = \int \frac{1}{\sqrt{1 - x^2}} \, dx
\]

We have

\( e^y = \arcsin(x) + C \)

Taking the ln of both sides, the general solution is

\( y = \ln(\arcsin(x) + C) \)

Notice that at least when we multiply through by \( e^y \) we do not lose any obvious solutions.

Problem 2. Find a solution to the initial value problem:

\[
\frac{dy}{dx} = \sqrt{1 - y^2} \sec^2(x)
\]

With initial value \( y(0) = 0 \).

Solution 2.

This is separable again. If we divide both sides by \( \sqrt{1 - y^2} \) and integrate we have

\[
\int \frac{1}{\sqrt{1 - y^2}} \, dy = \int \sec^2(x) \, dx
\]

So \( \arcsin(y) = \tan(x) + C \). We solve for \( C \) using our initial value, \( y(0) = 0 \).

\[
\arcsin(0) = \tan(0) + C
\]
So $C = 0$ works. If we take the sin of both sides the solution to our initial value problem is

$$y = \sin(\tan(x))$$

Notice that when we divide through by $\sqrt{1 - y^2}$ we make the assumption that $y \neq \pm 1$. These would both be valid solutions for the general solution, but not for our initial value problem.

**Problem 3.** Find a solution to the initial value problem:

$$\frac{dy}{dx} = (1 + y^2)e^x$$

With initial value $y(0) = 0$.

**Solution 3.**

This is separable so we divide by $1 + y^2$ and integrate

$$\int \frac{1}{1 + y^2} dy = \int e^x dx$$

And we have $\arctan(y) = e^x + C$. Solving for $C$

$$\arctan(0) = e^0 + C$$

So $C = -1$ works. Applying tan to both sides a solution to our initial value problem is

$$\tan(e^x - 1)$$

Notice that if we were solving for the general solution, when we divide through by $1 + y^2$ we do not lose any obvious solutions because it is strictly positive in $\mathbb{R}$.

**Problem 4.** Find the general solution to the differential equation (for $x \neq 0$):

$$x \frac{dy}{dx} = -y + x$$

**Solution 4.**

Unfortunately this is not separable, but we can put it in a recognizable form

$$\frac{dy}{dx} + \frac{1}{x}y = 1$$

Now we have to find our integrating factor which we know will be $e^{A(x)}$ where $A(x) = \int \frac{1}{x} dx = \ln(x)$. So our integrating factor is $e^{\ln(x)} = x$. What this tells us is that our equation was actually already in the form we desired:

$$\frac{d}{dx}(xy) = x$$
Integrating both sides we have
\[ xy = \frac{x^2}{2} + C \]
Dividing through by \( x \) the general solution is
\[ y = \frac{x^2 + C}{2x} \]
Notice that when we divide through by \( x \) here we use the assumption that \( x \neq 0 \).

**Problem 5.** Find the general solution to the differential equation
\[ \frac{1}{2x} \frac{dy}{dx} = y + e^{x^2} \]

**Solution 5.**

Once again we can get this into a recognizable form
\[ \frac{dy}{dx} - 2xy = 2xe^{x^2} \]
Now our integrating factor is \( e^{A(x)} \) where \( A(x) = \int -2x \, dx = -x^2 \). Multiplying through we have
\[ e^{-x^2} \frac{dy}{dx} - 2xe^{-2x}y = 2x \]
As usual this is of the form
\[ \frac{d}{dx} (e^{-x^2} y) = 2x \]
Integrating we have
\[ e^{-x^2} y = x^2 + C \]
Multiplying both sides by \( e^{x^2} \)
\[ y = e^{x^2} x^2 + Ce^{x^2} \]

**Problem 6.** Find a solution to the initial value problem
\[ \cos(x) \frac{dy}{dx} = 1 - \sin(x)y \]
With initial value \( y(0) = 1 \).

**Solution 6.**

We can put this in a recognizable form
\[ \frac{dy}{dx} + \tan(x)y = \sec(x) \]
Our integrating factor is then $e^\int \tan(x) \, dx = e^{-\ln(\cos(x))} = \sec(x)$. Multiplying through we have

$$\sec(x) \frac{dy}{dx} + \sec(x) \tan(x) y = \sec^2(x)$$

As usual this can be rewritten as

$$\frac{d}{dx} (\sec(x) y) = \sec^2(x)$$

Integrating both sides we have

$$\sec(x) y = \tan(x) + C$$

Solving for our initial value

$$\sec(0)(1) = \tan(0) + C$$

So $C = 1$ and the solution to our initial value problem is

$$y = \frac{\tan(x)}{\sec(x)} + \frac{1}{\sec(x)} = \sin(x) + \cos(x)$$

Notice the general solution is $y = \sin(x) + C \cos(x)$.