1. Introduction

Let $k \geq 2$ be an even integer. For $z$ in the complex upper half-plane we let $\Lambda_z \subset \mathbb{C}$ be the lattice $\{az + b | a, b \in \mathbb{Z}\}$, and define the function $E_\theta(k, z)$ by the absolutely convergent sum

$$E_\theta(k, z) = \sum_{(\lambda_1, \lambda_2, \lambda_3) \in \Lambda_z^3, \lambda_i \neq 0, \Sigma \lambda_i = 0} (\lambda_1 \lambda_2 \lambda_3)^{-k}.$$ 

If we let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be elements of $\mathbb{Z}^3$, we may rewrite this as

$$E_\theta(k, z) = \sum_{\Sigma a_i = 0} \sum_{\Sigma b_i = 0} \prod_{j=1}^{3} (a_j z + b_j)^{-k},$$

where the $'$ on the second sum denotes that we do not allow $a_i$ and $b_i$ to both be zero. It may be seen that $E_\theta(k, z) \in S_{3k}$, where $S_l$ is the space of modular forms of weight $l$ for $SL_2(\mathbb{Z})$. We denote the rational subspace of forms with rational Fourier coefficients by $S_l(\mathbb{Q})$. We shall prove the following result:

**Proposition 1.** We have $\pi^{-3k} E_\theta(k, z) \in S_{3k}(\mathbb{Q})$. Moreover, for any $k$ and $n$ there is an algorithm that calculates the first $n$ Fourier coefficients of $E_\theta(k, z)$.

Applying this to the constant term of $E_\theta(k, z)$, we recover the fact that

$$\sum_{\Sigma b_i = 0} (b_1 b_2 b_3)^{-k} \in \pi^{3k} \mathbb{Q}^\times.$$ 

These Eisenstein series are essentially equal to a Feynman graph integral of the Green’s function of the operator $(\partial / \partial z)^k$ on a complex torus, where the graph is taken to be the $\theta$-graph consisting of two points joined by three edges. It should be possible to prove Proposition 1 for the series associated to any graph.

2. Proof of Proposition 1

Because $S_{3k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = S_{3k}$, we may prove that $\pi^{-3k} E_\theta(k, z) \in S_{3k}(\mathbb{Q})$ by showing that all of its nonzero Fourier coefficients are rational. The $n$th Fourier coefficient of $E_\theta(k, z)$ is equal to

$$\sum_{\Sigma a_i = 0} \int_0^1 \sum_{\Sigma b_i = 0} \prod_{j=1}^{3} (a_j z + b_j)^{-k} e^{-2\pi i n x} dx.$$
and the following proposition implies that this lies in $\pi^3 \mathbb{Q}$. Moreover, it will be seen how to calculate this coefficient in the course of the proof.

**Proposition 2.** For any $n \geq 1$ and $a \neq 0$, the expression

$$I(n, a) = \int_0^1 \sum_{\Sigma b_j = 0}^{\prime} \prod_{j=1}^{3} (a_j x + b_j + ia_j y)^{-k} e^{-2\pi inx} dx$$

lies in $\pi^3 e^{-2\pi ny} \mathbb{Q}$. Moreover, for each fixed $n$ it is nonzero for only finitely many $a$.

**Proof.** We first consider the case in which $a_i \neq 0$ for all $i$. Define the function $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ by

$$f(x) = \prod_{j=1}^{3} (x_j + iy)^{-k}.$$

The Fourier transform of $f$, defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{2\pi i \langle x, \xi \rangle} dx,$$

is equal to

$$\hat{f}(\xi) = \begin{cases} (2\pi i)^{3k} \xi_1^{k-1} \xi_2^{k-1} \xi_3^{k-1} e^{-2\pi iy \sum \xi} & \text{if all } \xi_i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\Lambda_a$ to be the discrete subgroup $\{(b_1/a_1, b_2/a_2, b_3/a_3) | \Sigma b_i = 0\}$ of $\mathbb{R}^3$, and define $\Lambda_a f$ by

$$\Lambda_a f(x) = \sum_{\lambda \in \Lambda_a} f(x + \lambda).$$

We may rewrite the integral $I(a, n)$ as

$$I(a, n) = (a_1a_2a_3)^{-k} \int_0^1 \sum_{\Sigma b_j = 0}^{\prime} \prod_{j=1}^{3} (x + b_j/a_j + iy)^{-k} e^{-2\pi inx} dx$$

$$= (a_1a_2a_3)^{-k} \int_0^1 \Lambda_a f((x,x,x)) e^{-2\pi inx} dx.$$

We shall establish the proposition by examining the Fourier transform of $\Lambda_a f$. This function descends to $\mathbb{R}^3/\Lambda_a$, and the group of characters of this quotient is the closed subgroup $\Lambda_a^\perp$ of $\mathbb{R}^3$. The connected component of the identity in $\Lambda_a^\perp$ is a line which we give the parametrization $t(a_1, a_2, a_3)$, and the quotient by this line is isomorphic to $\mathbb{Z}^2$. The measure on $\Lambda_a^\perp$ dual to the Lebesgue measure on $\mathbb{R}^3/\Lambda_a$ is the Haar measure that is equal to $a_1a_2a_3 dt$ on the line $t(a_1, a_2, a_3)$. We denote this by $\mu_a$. By Fourier inversion we have
\[ \Lambda_\mathbf{a} f(x) = \int_{\Lambda_\mathbf{a}} \widehat{f}(\xi) e^{2\pi i (x, \xi)} d\mu_\mathbf{a}(\xi), \]

and substituting this into (2) gives

\[ I(\mathbf{a}, n) = (a_1a_2a_3)^{-k} \int_{\Lambda_\mathbf{a}} \widehat{f}(\xi) d\mu_\mathbf{a}(\xi). \]

We first show that there are only finitely many \( \mathbf{a} \) such that (3) is nonzero. We first observe that if (3) is to be nonzero, there must be \( \xi \in \Lambda_\mathbf{a} \) with \( \Sigma \xi = n \) and \( \widehat{f}(\xi) \neq 0 \), which implies that \( n \geq \xi_i > 0 \) for all \( i \). The condition \( \xi \in \Lambda_\mathbf{a} \) means that we must have

\[ \langle (\xi_1/a_1), \xi_2/a_2, /xi_3/a_3, \mathbf{b} \rangle \in \mathbb{Z} \]

for all \( \mathbf{b} \) with \( \Sigma b_i = 0 \), which is equivalent to requiring that all the differences \( \xi_i/a_i - \xi_j/a_j \) lie in \( \mathbb{Z} \).

If we have \( |a_i| > 2n \) for all \( i \), then the condition \( \xi_i < n \) means that \( |\xi_i/a_i - \xi_j/a_j| < 1 \) for all pairs \( i, j \), and this implies that \( \xi_1/a_1 = \xi_2/a_2 = \xi_3/a_3 \). However, two of the \( a_i \) must have opposite signs while the \( \xi_i \) are all positive, which is a contradiction. If we have \( |a_i|, |a_j| > 2n \) for distinct indices \( i \) and \( j \), we again have \( \xi_i/a_i = \xi_j/a_j \). This implies that \( a_i \) and \( a_j \) have the same sign, so the third entry of \( \mathbf{a} \) must have absolute value at least 4\( n \) which again leads to a contradiction.

We therefore see that the only contributions to (3) come from \( \mathbf{a} \) that have two indices of absolute value at most 2\( n \), and the condition \( \Sigma a_i = 0 \) implies that the number of such \( \mathbf{a} \) is finite.

To prove that \( I(n, \mathbf{a}) \in \pi^{3k} e^{-2\pi ny} \mathbb{Q} \), observe that it is equal to \((a_1a_2a_3)^{-k}\) times a finite sum of integrals

\[ \int \widehat{f}(\xi + t(a_1, a_2, a_3)) a_1a_2a_3 dt, \]

where \( \xi \) ranges over a set of representatives for the connected components of \( \Lambda_\mathbf{a} \) with \( \Sigma \xi = n \) that meet the positive quadrant. If the line \( \xi + t(a_1, a_2, a_3) \) lies in the positive quadrant for \( t \in [a, b] \), substituting the formula for \( \widehat{f} \) gives

\[ \int \widehat{f}(\xi + t(a_1, a_2, a_3)) a_1a_2a_3 dt = (2\pi i)^{3k} e^{-2\pi ny} \int_a^b \prod_{j=1}^3 (\xi_j + ta_j)^{k-1} dt. \]

The remaining integral is of a polynomial with rational coefficients over a rational interval, and hence is rational. This completes the proof in the case where all \( a_i \) are nonzero.

We now consider the case in which \( a_i = 0 \) for some \( i \). We may assume without loss of generality that \( a_3 = 0 \), which imposes the condition \( b_3 \neq 0 \) on \( \mathbf{b} \). We begin by fixing \( b_3 \) and considering the integrals

\[ I(n, \mathbf{a}, b_3) = \int_0^1 \sum_{b_1 + b_2 = -b_3} (a_1x + a_1iy + b_1)^{-k}(a_2x + a_2iy + b_2)^{-k}b_3^{-k} e^{-2\piinz} dx. \]
As we have $a_2 = -a_1$ and $b_2 = -b_1 - b_3$, we may rewrite this as

$$I(n, a, b_3) = a_1^{-2k}b_3^{-k} \int_0^1 \sum_{b_1} (x + iy + b_1/a_1)^{-k}(x + iy + b_1/a_1 + b_3/a_1)^{-k}e^{-2\pi inx} dx.$$ 

Unfolding the integral gives

$$I(n, a, b_3) = a_1^{-2k}b_3^{-k} \int_{-\infty}^{\infty} \sum_{b_1 \in \mathbb{Z}/a_1 \mathbb{Z}} (x + iy + b_1/a_1)^{-k}(x + iy + b_1/a_1 + b_3/a_1)^{-k}e^{-2\pi inx} dx$$

$$= a_1^{-2k}b_3^{-k} \sum_{b_1 \in \mathbb{Z}/a_1 \mathbb{Z}} e^{-2\pi inb_1/a_1} \int_{-\infty}^{\infty} (x + iy)^{-k}(x + iy + b_3/a_1)^{-k}e^{-2\pi inx} dx.$$ 

The sum over $b_1$ vanishes unless $a_1|n$, which proves that $I(n, a, b_3)$, and hence $I(n, a)$, vanish for all but finitely many $a$.

If $a_1|n$, we have

$$I(n, a, b_3) = a_1^{-2k+1}b_3^{-k} \int_{-\infty}^{\infty} (x + iy)^{-k}(x + iy + b_3/a_1)^{-k}e^{-2\pi inx} dx.$$ 

We shall calculate this integral by moving the contour into the lower half-plane. The integrand $(w + iy)^{-k}(w + iy + b_3/a_1)^{-k}e^{-2\pi inw}$ has poles at $-iy$ and $-iy - b_3/a_1$, and the residues there are equal to

$$\left( \frac{\partial}{\partial w} \right)^{-k-1} (w + iy + b_3/a_1)^{-k}e^{-2\pi inw} \bigg|_{w = -iy} = e^{-2\pi ny} \sum_{j=0}^{k-1} c(j)(2\pi i)^j b_3^{-2k+1+j}$$

and

$$\left( \frac{\partial}{\partial w} \right)^{-k-1} (w + iy)^{-k}e^{-2\pi inw} \bigg|_{w = -iy - b_3/a_1} = e^{-2\pi ny} \sum_{j=0}^{k-1} d(j)(2\pi i)^j b_3^{-2k+1+j}$$

respectively, where the coefficients $c(j)$ and $d(j)$ are rational numbers that depend on $k, a_1,$ and $j$, but not $b_3$. We therefore have

$$I(n, a, b_3) = a_1^{-2k+1}e^{-2\pi ny} \sum_{j=0}^{k-1} (c(j) + d(j))(2\pi i)^j b_3^{-3k+1+j}.$$ 

Summing over $b_3 \neq 0$ now gives

$$I(n, a) = a_1^{-2k+1}e^{-2\pi ny} \sum_{0 \leq j \leq k-1 \atop j \text{ odd}} (c(j) + d(j))(2\pi i)^j 2\zeta(3k - j - 1)$$

The fact that $\zeta(2k) \in \pi^{2k}\mathbb{Q}^\times$ for $k \geq 1$ completes the proof.