Corollary 3 of [1] is not known unconditionally, as cohomological automorphic forms on $GL_2$ over an imaginary quadratic field are not known to satisfy the Ramanujan conjecture. We shall briefly describe the reason for this and discuss what information Theorem 1 of [4] does give in the case of imaginary quadratic fields.

Let $K$ be an imaginary quadratic field with nontrivial automorphism $c$, and let $\pi$ be a cuspidal automorphic representation of $GL_2(A_K)$ with unitary central character $\omega$. Suppose that $\omega = \omega^c$ and that $\pi_\infty$ has Langlands parameter $W_C = \mathbb{C}^\times \to GL_2(\mathbb{C})$ given by $z \mapsto \text{diag}(z^{1-k}, z^{1-k})$ for some integer $k \geq 2$ (i.e. so that $\pi$ is any cohomological representation up to twist). It is then known (see Theorem 1.1 of [1]) that for any $\ell$ one may associate a continuous irreducible representation $\rho : \text{Gal}(\overline{K}/K) \to GL_2(\mathbb{Q}_\ell)$ to $\pi$ such that the characteristic polynomial of $\rho(\text{Frob}_v)$ agrees with the Hecke polynomial of $\pi_v$ at all places $v$ which do not divide $\ell$ and at which $K/\mathbb{Q}$, $\pi$, and $\pi^c$ are unramified. However, because $\rho$ is constructed via an $\ell$-adic limiting process, it is not known to arise from a motive and so is not known to be pure.

To construct $\rho$, one first makes a theta lift from $\pi$ to a holomorphic limit of discrete series representation $\Pi$ on $Sp_4/\mathbb{Q}$ as in [3]. Weissauer [6] has proven that if $\Pi'$ is a holomorphic discrete series representation of $Sp_4/\mathbb{Q}$ which is not a CAP representation, then one may associate a Galois representation to it which is pure and locally compatible with $\Pi'$ at all unramified places, so that $\Pi'$ satisfies Ramanujan wherever it is unramified. These results are not known for the limit of discrete series representation $\Pi$, and to associate a Galois representation to it one must apply techniques of Taylor [5] which are similar to those used by Deligne and Serre to associate Galois representations to classical weight 1 modular forms. These involve multiplying a holomorphic form in $\Pi$ by a well understood regular holomorphic form of large weight, applying the results of Weissauer and recovering $\rho$ from these products by an $\ell$-adic limiting process. Any Archimedean information about the Frobenius eigenvalues of $\rho$ is lost during this, and neither does one know that $\rho$ arises from a motive. As a result, the algebraic information we obtain about $\pi$ is insufficient to deduce Ramanujan for it.

We may still draw interesting conclusions from Theorem 1 of [4] in the imaginary quadratic case. Cohomological forms on $GL_2/K$ which are base changes from $\mathbb{Q}$ will satisfy Ramanujan, and so Theorem 1 establishes their equidistribution as their weight becomes large. Moreover, the experimental results of [2] suggest that all
but finitely many forms of fixed level and growing weight on $GL_2/K$ are obtained from base change and CM constructions, so that the question of whether a general cohomological form on $GL_2/K$ satisfies Ramanujan does not seem to matter in practice.

References


