Math 753: Week 1

Laurențiu Maxim

October 5, 2013
Chapter 1

Basics of Homotopy Theory

1.1 Homotopy Groups

Definition 1.1.1. For each $n \geq 0$ and $X$ a topological space with $x_0 \in X$, the $n$-th homotopy group of $X$ is defined as

$$\pi_n(X,x_0) = \{ f : (I^n, \partial I^n) \to (X,x_0) \} / \sim$$

where $\sim$ is the usual homotopy of maps.

Remark 1.1.2. Note that we have the following diagram of sets:

\[ \begin{array}{ccc}
(I^n, \partial I^n) & \xrightarrow{f} & (X,x_0) \\
\downarrow & & \downarrow \\
(I^n/\partial I^n, \partial I^n/\partial I^n) & \xrightarrow{g} & (S^n, s_0) \\
\end{array} \]

with $(I^n/\partial I^n, \partial I^n/\partial I^n) \simeq (S^n, s_0)$. So we can also define

$$\pi_n(X,x_0) = \{ g : (S^n, s_0) \to (X,x_0) \} / \sim .$$

Remark 1.1.3. If $n = 0$, then $\pi_0(X)$ is the set of connected components of $X$. Indeed, we have $I^0 = \text{pt}$ and $\partial I^0 = \emptyset$, so $\pi_0(X)$ consists of homotopy classes of maps from a point into the space $X$.

Now we will prove several results analogous to the case $n = 1$, which corresponds to the fundamental group.

Proposition 1.1.4. If $n \geq 1$, then $\pi_n(X,x_0)$ is a group with respect to the operation $+$ defined as:

$$(f + g)(s_1, s_2, \ldots, s_n) = \begin{cases} 
  f(2s_1, s_2, \ldots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\
  g(2s_1 - 1, s_2, \ldots, s_n) & \frac{1}{2} \leq s_1 \leq 1 
\end{cases}$$
Proof. First note that since only the first coordinate is involved in this operation, the same argument used to prove that $\pi_1$ is a group is valid here as well. Then the identity element is the constant map taking all of $I^n$ to $x_0$ and the inverse element is given by

$$-f(s_1, s_2, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n).$$

Proposition 1.1.5. If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.

Intuitively, since the $+$ operation only involves the first coordinate, if $n \geq 2$, there is enough space to “slide $f$ past $g$”.

Proof. Let $n \geq 2$ and let $f, g \in \pi_n(X, x_0)$. We wish to show $f + g \simeq g + f$. Consider the following figures:

We first shrink the domains of $f$ and $g$ to smaller cubes inside $I^n$ and map the remaining region to the base point $x_0$. Note that this is possible since both $f$ and $g$ map to $x_0$ on the boundaries, so the resulting map is continuous. Then there is enough room to slide $f$ past $g$ inside $I^n$. We then enlarge the domains of $f$ and $g$ back to their original size and get $g + f$. So we have constructed a homotopy between $f + g$ and $g + f$ and hence $\pi_n(X, x_0)$ is abelian.

Remark 1.1.6. If we view $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, then we have the following visual representation of $f + g$ (one can see this by collapsing boundaries in the above cube interpretation).
Next recall that if $X$ is path-connected and $x_0, x_1 \in X$, then there is an isomorphism

$$\beta_\gamma : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

where $\gamma$ is a path from $x_0$ to $x_1$, i.e., $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. The isomorphism $\beta_\gamma$ is given by

$$\beta_\gamma([f]) = [\gamma^{-1} \cdot f \cdot \gamma]$$

for any $[f] \in \pi_1(X, x_0)$.

We next show a similar fact holds for all $n \geq 1$.

**Proposition 1.1.7.** If $n \geq 1$ and $X$ is path-connected, then there is an isomorphism $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by

$$\beta_\gamma([f]) = [\gamma \cdot f],$$

where $\gamma$ is a path in $X$ from $x_1$ to $x_0$, and $\gamma \cdot f$ is constructed by first shrinking the domain of $f$ to a smaller cube inside $I^n$, and then inserting the path $\gamma$ radially from $x_1$ to $x_0$ on the boundaries of these cubes.

**Proof.** It is easy to check that the following properties hold:

1. $\gamma \cdot (f + g) \simeq \gamma \cdot f + \gamma \cdot g$
2. $(\gamma \cdot \eta) \cdot f \simeq \gamma \cdot (\eta \cdot f)$, for $\eta$ a path from $x_0$ to $x_1$
3. $c_{x_0} \cdot f \simeq f$, where $c_{x_0}$ denotes the constant path based at $x_0$.

4. $\beta_\gamma$ is well-defined with respect to homotopies of $\gamma$ or $f$.

Note that (1) implies that $\beta_\gamma$ is a group homomorphism, while (2) and (3) show that $\beta_\gamma$ is invertible. Indeed, if $\gamma(t) = \gamma(1-t)$, then $\beta_\gamma^{-1} = \beta_\gamma$. □

So, as in the case $n = 1$, if the space $X$ is path-connected, then $\pi_n$ is independent of the choice of base point. Further, if $x_0 = x_1$, then (2) and (3) also imply that $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$:

$$
\pi_1 \times \pi_n \to \pi_n \\
(\gamma, [f]) \mapsto [\gamma \cdot f]
$$

**Definition 1.1.8.** We say $X$ is an abelian space if $\pi_1$ acts trivially on $\pi_n$ for all $n \geq 1$.

In particular, this means $\pi_1$ is abelian, since the action of $\pi_1$ on $\pi_1$ is by inner-automorphisms, which must all be trivial.

We next show that $\pi_n$ is a functor.

**Proposition 1.1.9.** A map $\phi : X \to Y$ induces group homomorphisms $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$, for all $n \geq 1$.

**Proof.** First note that, if $f \simeq g$, then $\phi \circ f \simeq \phi \circ g$. Indeed, if $\psi_t$ is a homotopy between $f$ and $g$, then $\phi \circ \psi_t$ is a homotopy between $\phi \circ f$ and $\phi \circ g$. So $\phi_*$ is well-defined. Moreover, from the definition of the group operation on $\pi_n$, it is clear that we have $\phi \circ (f + g) = (\phi \circ f) + (\phi \circ g)$. So $\phi_*([f + g]) = \phi_*([f]) + \phi_*([g])$. Hence $\phi_*$ is a group homomorphism. □

The following is a consequence of the definition of the above induced homomorphisms:

**Proposition 1.1.10.** The homomorphisms induced by $\phi : X \to Y$ on higher homotopy groups satisfy the following two properties:

1. $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.

2. $(id_X)_* = id_{\pi_n(X, x_0)}$.

We thus have the following important consequence:

**Corollary 1.1.11.** If $\phi : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ is an isomorphism, for all $n \geq 1$.

**Example 1.1.12.** Consider $\mathbb{R}^n$ (or any contractible space). We have $\pi_i(\mathbb{R}^n) = 0$ for all $i \geq 1$, since $\mathbb{R}^n$ is homotopy equivalent to a point.

The following result is very useful for computations:

**Proposition 1.1.13.** If $p : \tilde{X} \to X$ is a covering map, then $p_* : \pi_n(\tilde{X}, \tilde{x}) \to \pi_n(X, p(\tilde{x}))$ is an isomorphism for all $n \geq 2$. 

4
Proof. First we claim $p_*$ is surjective. Let $x = p(\tilde{x})$ and consider $f : (S^n, s_0) \to (X, x)$. Since $n \geq 2$, we have that $\pi_1(S^n) = 0$, so $f_*\pi_1(S^n, s_0) = 0 \subset p_*\pi_1(\tilde{X}, \tilde{x})$. So $f$ admits a lift, i.e., there is $\tilde{f} : (S^n, s_0) \to (\tilde{X}, \tilde{x})$ such that $p \circ \tilde{f} = f$. Then $[f] = [p \circ \tilde{f}] = p_*([\tilde{f}])$. So $p_*$ is surjective.

\begin{center}
\begin{tikzpicture}
  \node (S) at (0, 0) {$(S^n, s_0)$};
  \node (X) at (2, 0) {$(X, x)$};
  \node (Xtilde) at (2, -2) {$(\tilde{X}, \tilde{x})$};

  \draw[->] (S) to node[above] {$f$} (X);
  \draw[->] (S) to node[left] {$p$} (Xtilde);
  \draw[->] (X) to node[left] {$f$} (Xtilde);
\end{tikzpicture}
\end{center}

Next, we show that $p_*$ is injective. Suppose $[\tilde{f}] \in \ker p_*$. So $p_*([\tilde{f}]) = [p \circ \tilde{f}] = 0$. Let $p \circ \tilde{f} = f$. Then $f \simeq c_x$ via some homotopy $\phi_t : (S^n, s_0) \to (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_x$. Again, by the lifting criterion, there is a unique $\tilde{\phi}_t : (S^n, s_0) \to (\tilde{X}, \tilde{x})$ with $p \circ \tilde{\phi}_t = \phi_t$.

\begin{center}
\begin{tikzpicture}
  \node (S) at (0, 0) {$(S^n, s_0)$};
  \node (X) at (2, 0) {$(X, x)$};
  \node (Xtilde) at (2, -2) {$(\tilde{X}, \tilde{x})$};

  \draw[->] (S) to node[above] {$\phi_t$} (X);
  \draw[->] (S) to node[left] {$p$} (Xtilde);
  \draw[->] (X) to node[left] {$\tilde{\phi}_t$} (Xtilde);
\end{tikzpicture}
\end{center}

Then we have $p \circ \tilde{\phi}_1 = \phi_1 = f$ and $p \circ \tilde{\phi}_0 = \phi_0 = c_x$, so by the uniqueness of lifts, we must have $\tilde{\phi}_1 = \tilde{f}$ and $\tilde{\phi}_0 = c_{\tilde{x}}$. Then $\tilde{\phi}_t$ is a homotopy between $\tilde{f}$ and $c_{\tilde{x}}$. So $[\tilde{f}] = 0$. Thus $p_*$ is injective. \qed

Example 1.1.14. Consider $S^1$ with its universal covering map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$. We already know $\pi_1(S^1) = \mathbb{Z}$. If $n \geq 2$, Proposition 1.1.13 yields that $\pi_n(S^1) = \pi_n(\mathbb{R}) = 0$.

Example 1.1.15. Consider $T^n = S^1 \times S^1 \times \ldots \times S^1$, the $n$-torus. We have $\pi_1(T^n) = \mathbb{Z}^n$. By using the universal covering map $p : \mathbb{R}^n \to T^n$, we have by Proposition 1.1.13 that $\pi_i(T^n) = \pi_i(\mathbb{R}^n) = 0$ for $i \geq 2$.

Definition 1.1.16. If $\pi_n(X) = 0$ for all $n \geq 2$, the space $X$ is called aspherical.

Proposition 1.1.17. Let $\{X_\alpha\}_\alpha$ be a collection of path-connected spaces. Then

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$$

for all $n$.

Proof. First note that a map $f : Y \to \prod_\alpha X_\alpha$ is a collection of maps $f_\alpha : Y \to X_\alpha$. For elements of $\pi_n$, take $Y = S^n$ (note that since all spaces are path-connected, we may drop the reference to base points). For homotopies, take $Y = S^n \times I$. \qed
Example 1.1.18. It is a natural question to find two spaces $X$ and $Y$ such that $\pi_n(X) \cong \pi_n(Y)$ for all $n$, but with $X$ and $Y$ not homotopy equivalent. Whitehead’s Theorem (to be discussed later on) states that if a map of CW complexes $f : X \to Y$ induces isomorphisms on all $\pi_n$, then $f$ is a homotopy equivalence. So we must find $X$ and $Y$ so that there is no continuous map $f : X \to Y$ inducing the isomorphisms on $\pi_n$’s. Consider $X = S^2 \times \mathbb{R}P^3$ and $Y = \mathbb{R}P^2 \times S^3$. Then $\pi_n(X) = \pi_n(S^2 \times \mathbb{R}P^3) = \pi_n(S^2) \times \pi_n(\mathbb{R}P^3)$. Since $S^3$ is a covering of $\mathbb{R}P^3$, for all $n \geq 2$ we have that $\pi_n(X) = \pi_n(S^2) \times \pi_n(S^3)$. We also have $\pi_1(X) = \pi_1(S^2) \times \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2$. Similarly, we have $\pi_n(Y) = \pi_n(\mathbb{R}P^2 \times S^3) = \pi_n(\mathbb{R}P^2) \times \pi_n(S^3)$. And since $S^2$ is a covering of $\mathbb{R}P^2$, for $n \geq 2$ we have that $\pi_n(Y) = \pi_n(S^2) \times \pi_n(S^3)$. Finally, $\pi_1(Y) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \mathbb{Z}/2$. So $\pi_n(X) = \pi_n(Y)$ for all $n$. By considering homology groups, however, we see that $X$ and $Y$ are not homotopy equivalent. Indeed, by the Künneth formula, we get that $H_5(X) = 0$ while $H_5(Y) = \mathbb{Z}/2$.

Just like there is a homomorphism $\pi_1(X) \to H_1(X)$, we can also construct homomorphisms $\pi_n(X) \to H_n(X)$ defined by $[f : S^n \to X] \mapsto f_*[S^n]$, where $[S^n]$ is the fundamental class of $S^n$. A very important result in homotopy theory is the following:

**Theorem 1.1.19. (Hurewicz)**

If $n \geq 2$ and $\pi_i(X) = 0$ for all $i < n$, then $H_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$.

Moreover, there is also a relative version of the Hurewicz theorem (see the next section for a definition of the relative homotopy groups), which can be used to prove the following:

**Corollary 1.1.20.** If $X$ and $Y$ are CW complexes with $\pi_1(X) = \pi_1(Y) = 0$, and a map $f : X \to Y$ induces isomorphisms on all integral homology groups $H_n$, then $f$ is a homotopy equivalence.