1. Relative homotopy groups

Let \( x_0 \in A \subset X \), and \( n \in \mathbb{Z} \) with \( n \geq 1 \). Let

\[
P^{n-1} = \{(s_1, \ldots, s_n) \in I^n : s_n = 0\}.
\]

Define \( J^{n-1} \) by

\[
J^{n-1} := \partial I^n - I^{n-1}.
\]

Then we define the \( n \)th homotopy group of the pair \((X, A)\) with basepoint \( x_0 \) to be

\[
\pi_n(X, A, x_0) = \{f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\}/\text{homotopy equivalence}.
\]

We have the following commutative triangle

\[
(I^n, \partial I^n, J^{n-1}) \xrightarrow{f} (X, A, x_0) \xrightarrow{g} (D^n, S^{n-1}, s_0)
\]

where \( g \) is obtained by collapsing \( J^{n-1} \).

**Special case:** \( n = 1 \). Then

\[
\pi_1(X, A, x_0) = \{f : (I, \{0, 1\}, \{1\}) \to (X, A, x_0)\}/\text{homotopy equivalence}.
\]

In words, \( \pi_1(X, A, x_0) \) is the set of paths that start anywhere in \( A \) and end at \( x_0 \) (up to homotopy equivalence). We cannot define a group structure on this set.

**Remark 1.** \( \pi_n(X, x_0, x_0) = \pi_n(X, x_0) \).

**Proposition 1.** For \( n \geq 2 \), \( \pi_n(X, A, x_0) \) is a group with the usual \(+\) operation as in the absolute case. Moreover, \( \pi_n(X, A, x_0) \) is abelian if \( n \geq 3 \).

**Proof.** The same as in the absolute case. \( \square \)

**Remark 2.** Maps of pairs \( \varphi : (X, A, x_0) \to (Y, B, y_0) \) induce (by composition) group homomorphisms

\[
\varphi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)
\]

\[
[f] \mapsto [\varphi \circ f].
\]

**Proposition 2.** The relative homotopy groups fit into a long exact sequence:

\[
\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0) \xrightarrow{1} 0
\]
where \(i_*\) and \(j_*\) are induced by inclusions of pairs/triples, \(i : A \hookrightarrow X, j : (X, x_0) \hookrightarrow (X, A)\), and \(\partial^n([f]) := [f]
_{|I^{n-1}}\).

Proof. Exercise.

**Remark 3.** Towards the end of the sequence exactness still makes sense even though no group structures are present.

**Lemma 1.** Let \([f] \in \pi_n(X, A, x_0)\). Then \([f] = 0\) if and only if \(f \simeq g\) so that \(\text{im}(g) \subseteq A\).

**Sketch of Proof.** It is clear that if \([f] = 0\) then \(f \simeq g\) with \(\text{im}(g) \subseteq A\), since \(f\) is homotopic to the constant map at \(x_0\). Conversely suppose that \(f \simeq g\) with \(\text{im}(g) \subseteq A\). Deform \(I^n\) to \(J^{n-1}\) by pushing \(I^{n-1}\) inside. Then we obtain \(g \simeq c_{x_0}\).

**2. (In)dependence of basepoints.**

**Proposition 3.** If \(A\) is path connected, then 

\[ \beta_\gamma : \pi_n(X, A, x_0) \to \pi_n(X, A, x_1) \]

is an isomorphism where \(\gamma\) is a path in \(A\) from \(x_0\) to \(x_1\).

**Proof.** Similar to the absolute case.

**Remark 4.** \(\pi_1(A)\) acts on \(\pi_n(X, A, x_0)\) (just take \(x_0 = x_1\)).

We say that \((X, A)\) is \(n\)-connected if \(\pi_i(X, A) = 0\) for all \(i \leq n\). We say that \(X\) is \(n\)-connected if \(\pi_i(X) = 0\) for all \(i \leq n\).

Observe that \(X\) is 0-connected if and only if \(X\) is connected and \(X\) is 1-connected if and only if \(X\) is simply connected.

**Hurewicz Theorem.** Given any connected topological space \(X\) there exists a map

\[ \pi_1(X) \to H_1(X) = \pi_1(X)^{ab} \]

\[ [f : S^1 \to X] \mapsto f_*([S^1]), \]

where \([S^1]\) denotes the fundamental class of \(S^1\). More generally there is a map of

\[ \pi_n(X) \to H_n(X) \]

\[ [f : S^n \to X] \mapsto f_*([S^n]). \]

**Theorem 1.** If \(n \geq 2\) and \(\pi_i(X) = 0\) for all \(i < n\) then \(H_i(X) = 0\) for all \(i < n\) and \(\pi_n(X) \cong H_n(X)\) (and onto on \(\pi_{n+1}\)).

**Proof.** Will be proven later.

**3. Homotopy groups of spheres.**

What is \(\pi_i(S^n)\)?

For \(i < n\), \(\pi_i(S^n) = 0\). For \(i = n, n + 1\), \(\pi_i(S^n) = \mathbb{Z}\).

For \(i = n\), \(f_*([S^n]) \mapsto \text{deg}(f) \cdot [S^n]\) is an isomorphism. To show that \(\pi_i(S^n) = 0\) for \(i < n\) use cellular approximation.

**Theorem 2.** Given \(f : X \to Y\) a map of CW complexes, one can find a cellular map \(g : X \to Y\) (i.e. \(g(X_i) \subseteq Y_i\) for all \(i\)) such that \(f \simeq g\). Similarly for maps of pairs.
Corollary 1. For $i < n$, $\pi_i(S^n) = 0$.

Proof. Can assume $f : S^i \to S^n$ is cellular, which implies that $f(S^i)i \subseteq (S^n)i = pt$, so $f$ is homotopic to a constant map. □

Corollary 2. Let $A \subseteq X$ be CW complexes and suppose all cells in $X - A$ have dimension greater than $n$. Then $\pi_i(X, A) = 0$ for all $i \leq n$.

Corollary 3. For all $i \leq n$, $\pi_i(X, X_n) = 0$.

From the above corollary and the long exact sequence of relative homotopy groups we obtain the following corollary:

Corollary 4. For all $i \leq n - 1$, $\pi_i(X) \cong \pi_i(X_n)$.

Proof of Corollary 2. For $[f] \in \pi_i(X, A)$, the map $f : (D^i, S^{i-1}) \to (X, A)$ is homotopic to $g$ such that $g(D^i) \subseteq X_i \subseteq A$ for all $i \leq n$. Thus $[f] = 0$ by Lemma 1. □

Suspension Theorem. Let $f : S^i \to S^n$ be a map. Consider its suspension $\Sigma f : \Sigma S^i \cong S^{i+1} \to \Sigma S^n \cong S^{n+1}$. The assignment

$$
\pi_i(S^n) \to \pi_{i+1}(S^{n+1})
$$

$[f] \mapsto [\Sigma f]$ is an isomorphism for $i < 2n - 1$ and onto for $i = 2n - 1$.

Corollary 5. We have $\pi_n(S^n)$ is either $\mathbb{Z}$ or a finite quotient of $\mathbb{Z}$, generated by the degree map.

Proof. By the Suspension Theorem

$$
\mathbb{Z} \cong \pi_1(S^1) \to \pi_2(S^2) \cong \ldots \cong \pi_n(S^n) \cong \ldots
$$

To get that $\pi_1(S^1) \to \pi_2(S^2)$ is an isomorphism, use the long exact sequence of $\pi_i$ for the Hopf fibration, or alternatively use Hurewicz. □

Recall that if $X$ is a finite CW complex, then $H_*(X)$ and $H^*(X)$ are finitely generated abelian groups. This is not true for homotopy groups.

Example 1. Let $X = S^1 \vee S^2$. Then $\pi_2(X) = \pi_2(\tilde{X})$, for $\tilde{X}$ the universal cover of $X$. Recall that the universal cover of $X$ is the real line, with spheres attached at the integer points, which is homotopy equivalent to $\bigvee_{k \in \mathbb{Z}} S^2_k$. We claim that $\pi_2 \left( \bigvee_{k \in \mathbb{Z}} S^2_k \right)$ is generated by inclusions of factors $[S^2_k \to \bigvee_{k \in \mathbb{Z}} S^2_k]$. Assuming this, $\pi_1(X) = \mathbb{Z}$, which implies that Deck($\tilde{X}/X$) = $\mathbb{Z}$. Since $\pi_1(X)$ acts on $\pi_2(X)$, we have that $\pi_2(X)$ is a $\mathbb{Z}[\pi_1]$-module. Now

$$
\mathbb{Z}[\pi_1] = \left\{ \sum_i n_i \gamma_i : n_i \in \mathbb{Z}, \gamma_i \in \pi_i \right\} \cong \mathbb{Z}\langle t \rangle \cong \mathbb{Z}[t, t^{-1}]
$$

(where this last isomorphism is obtained via the map $n \mapsto t^n$). Thus we see that $\pi_2(X)$ is a free $\mathbb{Z}[\pi_1]$-module of rank 1 with generator any of $[S^2_k \to \tilde{X}]$. So $\pi_2(X) = \mathbb{Z}[\pi_1] \cong \mathbb{Z}[t, t^{-1}]$ which is infinitely generated as a $\mathbb{Z}$-module by $\{t^k : k \in \mathbb{Z}\}$. 

Lemma 2. \( \pi_n(\bigvee_{\alpha} S^n_{\alpha}) \) is generated (over \( \mathbb{Z} \)) by the inclusions of factors \([S^n_{\alpha} \hookrightarrow \bigvee_{\alpha} S^n_{\alpha}]\).

Proof. If there are only finitely many \( S^n_{\alpha} \), then
\[
\pi_n\left( \bigvee_{\alpha} S^n_{\alpha} \right) \cong \pi_n\left( \prod_{\alpha} S^n_{\alpha} \right) \quad \text{(remains to be proven)}
\]
\[
\cong \prod_{\alpha} \pi_n(S^n_{\alpha})
\]
\[
\cong \bigoplus_{\alpha} \mathbb{Z}.
\]

To prove the first isomorphism, we claim that \( \bigvee_{\alpha} S^n_{\alpha} \) is the \( n \)-skeleton of \( \prod_{\alpha} S^n_{\alpha} \). Now \( S^n_{\alpha} = e^n_{\alpha} \cup e^n_{\alpha} \) and \( \bigvee_{\alpha} S^n_{\alpha} = e^n_{\alpha} \cup (e^n_{\alpha})_{\alpha} \). In \( \prod_{\alpha} S^n_{\alpha} \) there is one 0-cell, \( \prod_{\alpha} e^n_{\alpha} := e^n \) and \( n \)-cells of the form \( \bigcup_{\alpha} \left( \prod_{\beta \neq \alpha} e^n_{\beta} \right) \times e^n_{\alpha} \). Thus we see that \( \bigvee_{\alpha} S^n_{\alpha} \) is indeed the \( n \)-skeleton of \( \prod_{\alpha} S^n_{\alpha} \). So \( \prod_{\alpha} S^n_{\alpha} - \bigvee_{\alpha} S^n_{\alpha} \) only has cell of dimension at least \( 2n \). From Corollary 2, \((\prod_{\alpha} S^n_{\alpha}, \bigvee_{\alpha} S^n_{\alpha})\) is (\( 2n - 1 \))-connected. Since \( n \geq 2 \), and thus \( n < 2n - 1 \), we have that \( \pi_n(\bigvee_{\alpha} S^n_{\alpha}) = \pi_n(\prod_{\alpha} S^n_{\alpha}) \).

In general, any \( f : S^n \to \bigvee_{\alpha} S^n_{\alpha} \) has compact image, and thus touched only finitely many spheres, \( S^n_{\alpha} \). From this we obtain that \( \bigoplus_{\alpha} \pi_n(S^n_{\alpha}) \to \pi_n(\bigvee_{\alpha} S^n_{\alpha}) \) is an isomorphism. \( \square \)

Whitehead’s Theorem. If \( f : X \to Y \) is a map of CW complexes which induces isomorphisms on all \( \pi_n \), then \( f \) is a homotopy equivalence. Moreover, if \( f \) is an inclusion of a subcomplex, then \( X \) is a deformation retract of \( Y \).

Corollary 6. If \( f : X \to Y \) induces isomorphisms on \( H_*(-, \mathbb{Z}) \) and \( \pi_1(X) = \pi_1(Y) = 0 \), then \( f \) is a homotopy equivalence.

Before we can proceed, we need the following lemma:

Lemma 3. In homotopy theory “any map is an inclusion”. That is, any \( f : X \to Y \) can be factored as \( X \xleftarrow{i} M_f \xrightarrow{r} Y \), where \( r \) is a deformation retract.

Proof. Define the mapping cylinder of \( f \) by
\[
M_f := (X \times [0, 1]) \coprod Y/((x, 1) \sim f(x)).
\]
Note that \( X \subseteq M_f \) and \( Y \subseteq M_f \). Moreover, we have a deformation retract \( r : M_f \to Y \). Thus in the above theorem and corollary one can replace \( f \) by \( X \leftarrow M_f \), provided that \( M_f \) has a CW structure. If \( f \) is cellular then \( M_f \) has a CW structure (exercise). If \( f \) is not cellular then \( f \simeq g \) with \( g \) cellular, and one can work with \( X \leftarrow M_g \). \( \square \)

Proof of Corollary 6. From the above lemma, we can assume that \( f : X \to Y \) is an inclusion. From the long exact sequence of homology groups for \( (Y, X) \) we obtain that \( H_i(Y, X) = 0 \) for all \( i \). By the relative version of the Hurewicz Theorem, we get that \( \pi_i(Y, X) = 0 \) for all \( i \). The long exact sequence of relative homotopy groups for the pair \((Y, X)\) tells us that \( f_* : \pi_i(X) \to \pi_i(Y) \) is an isomorphism. Thus from Whitehead’s Theorem we see that \( f \) is a homotopy equivalence. \( \square \)

Remark/Example 1. The simply connected assumptions of the above corollary are necessary. Let \( X = S^1 \) and \( Y = (S^1 \vee S^n) \cup e^{n+1} \) where the attaching map \( \partial e^{n+1} = S^n \to S^1 \vee S^n \).
corresponds to the element $2t - 1 \in \mathbb{Z}[t, t^{-1}] \cong \pi_n(S^1 \vee S^n)$. So we have that $\pi_1(X) = \mathbb{Z} \neq 0$, $f_* : H_i(X) \rightarrow H_i(Y)$ is an isomorphism for all $i$, and $\pi_i(X) = 0$ for all $i > 1$. However,

$$
\pi_n(Y) \cong \pi_n(S^1 \vee S^n)/(2t - 1) \neq 0 \\
\cong \mathbb{Z}[t, t^{-1}]/(2t - 1) \\
\cong \mathbb{Z}[\frac{1}{2}].
$$

**Exercise:** Find spaces $X$ and $Y$ with the same homotopy groups so the isomorphisms cannot be induced by any map.