1 Whitehead’s theorem.

Statement: If \( f : X \to Y \) is a map of CW complexes inducing isomorphisms on all homotopy groups, then \( f \) is a homotopy equivalence. Moreover, if \( f \) is the inclusion of a subcomplex \( X \) in \( Y \), then there is a deformation retract of \( Y \) onto \( X \).

For future reference, we make the following definition:

Definition: \( f : X \to Y \) is a Weak Homotopy Equivalence (WHE) if it induces isomorphisms on all homotopy groups \( \pi_n \).

Notice that a homotopy equivalence is a weak homotopy equivalence.

Using the definition of Weak Homotopic Equivalence, we paraphrase the statement of Whitehead’s theorem as:

If \( f : X \to Y \) is a weak homotopy equivalences on CW complexes then \( f \) is a homotopy equivalence.

In order to prove Whitehead’s theorem, we will first recall the homotopy extension property and state and prove the Compression lemma.

**Homotopy Extension Property (HEP):**

Given a pair \((X,A)\) and maps \( F_0 : X \to Y \), a homotopy \( f_t : A \to Y \) such that \( f_0 = F_0|_A \), we say that \((X,A)\) has (HEP) if there is a homotopy \( F_t : X \to Y \) extending \( f_t \) and \( F_0 \). In other words, \((X,A)\) has homotopy extension property if any map \( X \times \{0\} \cup A \times I \to Y \) extends to a map \( X \times I \to Y \).

Question: Does the pair \(([0,1], \{\frac{1}{n}\}_{n\in\mathbb{N}})\) have the homotopic extension property?

Answer: No.

**Compression Lemma:** If \((X,A)\) is a CW pair and \((Y,B)\) is a pair with \( B \neq \emptyset \) so that for each \( n \) for which \( X \setminus A \) has \( n \)-cells, \( \pi_n(Y,B,b_0) = 0 \) for all \( b_0 \in B \), then any map \( f : (X,A) \to (Y,B) \) is homotopic to \( f' : X \to B \) fixing \( A \). (i.e. \( f'|_A = f|_A \)).

To prove the compression lemma, we will first prove the following proposition:

**Proposition:** Any CW pair has the homotopy extension property. In fact, for every CW pair \((X,A)\), there is a deformation retract \( r : X \times I \to X \times \{0\} \cup A \times I \) and we can then define \( X \times I \to Y \) by \( X \times I \to X \times \{0\} \cup A \times I \to Y \).

**Proof:** We have that \( D^n \times I \xrightarrow{r} X \times \{0\} \cup A \times I \) (where \( r \) is a deformation retraction). For every \( n \), looking at the \( n \)-skeleton \( X_n \) and considering the pair \((X_n, A_n \cup X_{n-1})\), we get that \( X_n \times I = [X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I] \cup D^n \times I \) where the cylinders \( D^n \times I \) corresponding to \( n \)-cells \( D^n \) in \( X \setminus A \) are glued along \( D^n \times (\{0\} \cup X_{n-1} \times I) \) to the pieces \([X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I] \). By deforming these cylinders \( D^n \times I \) we get a deformation retraction \( r_n : X_n \times I \to X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I \). Concatenating these deformation retractions by performing \( r_n \) over \([1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}]\), we get a deformation retraction of \( X \times I \) onto \( X \times \{0\} \cup A \times I \). Continuity follows since CW complexes have the weak topology with respect to their skeleta, so a map is continuous iff its restriction to each skeleton is continuous.
Proof of the Compression Lemma: We will prove this by induction on \( n \).

Assume \( f(X_{k-1} \cup A) \subseteq B \). Let \( e^k \) be a \( k \)-cell in \( X \setminus A \). Look at its characteristic map \( \alpha : (D^k, S^k) \to (X_k, X_{k-1} \cup A) \). Regard \( \alpha \) as an element \( [\alpha] \in \pi_k(X_k, X_{k-1} \cup A) \). Looking at \( f_*[\alpha] = [f \circ \alpha] \in \pi_k(Y, B) \) and using the fact that \( \pi_k(Y, B) = 0 \) by our hypothesis (and since \( e^k \in X \setminus A \)), we get a homotopy \( H : (D^k, S^k-1) \times I \to (Y, B) \) such that \( H_0 = f \circ \alpha \) and \( \text{Im}H_1 \subseteq B \).

Performing this process for all the \( k \) cells in \( X \setminus A \) simultaneously, we get a homotopy \( H_k \) from \( f \) to \( f' \) such that \( f'(X_k \cup A) \subseteq B \). Using the homotopy extension property, regard this as a homotopy on all of \( X \). We hence get \( f \simeq f_1 \), such that \( f_1(X_1 \cup A) \subseteq B \), \( f_1 \simeq f_2 \) such that \( f_2(X_2 \cup B) \), and so on.

Define \( H : X \times I \to Y \) as \( H = H_1 \) on \([1 - \frac{1}{2^n} 1 - \frac{1}{n}] \). \( H \) is continuous by CW topology, thus giving us the required homotopy. \( \Box \)

Proof of Whitehead’s Theorem: We can assume \( f \) is an inclusion (by using cellular approximation and the mapping cylinder \( M_f \)). Let \((Y, X)\) be a CW pair. By assumption and the long exact sequence of the pair, we have that \( \pi_n(Y, X) = 0 \) for all \( n \). By applying the compression lemma to the identity map of \((Y, X)\), we get the desired deformation retract \( r : Y \to X \). \( \Box \)

Example: Let \( X = \mathbb{R}P^2 \), \( Y = S^2 \times \mathbb{R}P^\infty \). We know that \( \pi_1(X) = \mathbb{Z}/2\mathbb{Z} = \pi_1(Y) \), \( \pi_n(\mathbb{R}P^2) = \pi_n(S^2) \) for all \( n \geq 2 \).

Also note that, \( \pi_n(S^2 \times \mathbb{R}P^\infty) = \pi_n(S^2) \times \pi_n(\mathbb{R}P^\infty) \), and that \( \pi_n(\mathbb{R}P^\infty) = \pi_n(S^\infty) = 0 \), since \( S^\infty \) is contractible. We hence have that \( \pi_n(S^2 \times \mathbb{R}P^\infty) = \pi_n(S^2) \)

So \( \pi_n(X) = \pi_n(Y) \) for all \( n \geq 1 \). However, \( X \not\simeq Y \) since their second homology groups are unequal, as \( H_2(\mathbb{R}P^2) = 0 \) and \( H_2(S^2 \times \mathbb{R}P^\infty) \neq 0 \). \( \Box \)

Theorem: Weak homotopy equivalence induce isomorphisms on \( H_*(\_, G) \) and \( H^*(\_, G) \) for any coefficient ring \( G \).

Proof in the simply connected case:

Let \( f : X \to Y \) be a weak homotopy equivalence. By the Universal Coefficient theorem, it is sufficient to show that \( f \) induces isomorphisms on integral homology \( H_*(\_ , \mathbb{Z}) \). We can also assume \( f \) is the inclusion map. Since \( f \) is a weak homotopy, via the long exact sequence, we have \( \pi_n(Y, X) = 0 \) for all \( n \). Combining this fact along with the fact that the fundamental group of \( X \) is 0, by using the Hurewicz theorem we get that \( H_n(Y, X) = 0 \) for all \( n \). Hence, using the long exact sequence for homology, we get \( H_n(X) \cong H_n(Y) \). \( \Box \)

Exercise: Show that any finitely generated (abelian when \( n \geq 2 \)) group can be realized as the \( n^{th} \) homotopy group for some space \( X \). \( \Box \)
2 Cellular approximation.

**Cellular Approximation:** If \( f : X \to Y \) is a continuous map of CW complexes then \( f \) has a cellular approximation \( f' : X \to Y \), i.e. \( f \simeq f' \) such that \( f(X_n) \subseteq Y_n \) for all \( n \geq 0 \). Moreover, if \( f \) is already cellular on some subcomplex \( A \subseteq X \), then we can perform cellular approximation relative to \( A \), i.e. \( f|_A' = f|_A \).

The proof of this relies on the technical lemma which states that if \( f : X \cup e^n \to Y \cup e^k \) (where \( e^n, e^k \) are \( n \) cells and \( k \) cells respectively) such that \( f(X) \subseteq Y \) and \( f|_X \) is cellular and if \( n \leq k \), then \( f \simeq f' \), with \( \text{Im}(f') \subseteq Y \). The technical lemma is used along with induction on skeleta to prove the above result on cellular approximation.

Remark: If \( X \) and \( Y \) are points in the statement of the above lemma then we get that \( S^n \hookrightarrow S^k \) can be homotoped into the constant map \( S^n \to \{ s_0 \} \) for some point \( s_0 \in S^k \).

Relative cellular approximation:

Any map \( f : (X, A) \to (Y, B) \) of CW pairs has a cellular approximation by a homotopy through such maps of pairs.

**Proof:** First use cellular approximation for \( f|_A : A \to B, f|_A \simeq f' \) where \( f' : A \to B \) is a cellular map. Using the Homotopy Extension Property, we can regard \( H \) as a homotopy on all of \( X \), so we get a map \( f' : X \to Y \) such that \( f'|_A \) is a cellular map. By the second statement of cellular approximation, \( f' \simeq f'' \) where \( f'' : X \to Y \) is a cellular map \( f'|_A = f''|_A \).

3 CW approximation

We are going to show that given any space \( X \) there exists a (unique up to homotopy) CW complex \( Z \), with a weak homotopic equivalence \( f : Z \to X \). Such a \( Z \) is called a CW approximation of \( X \).

**Definition:** Given a pair \( (X, A) \) with \( \emptyset \neq A \subseteq X \), where \( A \) is a CW complex, an \( n \)-connected model of \( (X, A) \) is an \( n \)-connected CW pair \( (Z, A) \), together with a map \( f : Z \to X \), \( f|_A = id|_A \) so that \( f_* : \pi_i(Z) \to \pi_i(X) \) is an isomorphism for \( i > n \) and is injective when \( i = n \).

**Remark:** If such models exist, we can take \( A \) to be some point on \( X \), let \( n = 0 \) and we get a cellular approximation \( Z \) of \( X \).

**Theorem:** Such \( n \)-connected models \( (Z, A) \) of a pair \( (X, A) \) (with \( A \) is a CW complex) exist. Moreover, \( Z \) can be obtained from \( A \) by attaching cells of dimension greater than \( n \). (Note that from cellular approximation we then have that \( \pi_i(Z, A) = 0 \) for \( i \leq n \)).

We will prove this theorem after proving two following corollaries:

**Corollary:** Any pair of spaces \( (X, X_0) \) has a CW approximation \( (Z, Z_0) \).

**Proof:** Let \( f_0 : Z_0 \to X_0 \) be a CW approximation of \( X_0 \). Consider the map \( g : Z_0 \to X \) defined by the composition of \( f_0 \) and the inclusion map \( X_0 \hookrightarrow X \). Consider the mapping
cylinder $M_g$ of $g$, i.e. $M_g = (Z_0 \times I) \sqcup X/(z_0, 1) \sim g(z_0)$. We hence get the sequence of maps $Z_0 \hookrightarrow M_g \rightarrow X$ where the map $M_g \rightarrow X$ is a deformation retract.

Now, let $(Z, Z_0)$ be a 0-connected CW model of $(M_g, Z_0)$. Consider the triangle:

$$(Z, Z_0) \longrightarrow (M_g, Z_0) \downarrow (X, X_0).$$

This gives us a map $f : Z \rightarrow X$ that is obtained by composing the weak homotopy equivalence $Z \rightarrow M_g$ and the deformation retract (hence homotopy equivalence) $M_g \rightarrow X$. In other words, $f$ is a weak homotopy equivalence and $f |_{Z_0} = f_0$, thus proving the result.

Corollary: For each $n$-connected CW pair $(X, A)$ there is a CW pair $(Z, A)$ that is homotopy equivalent to $(X, A)$ relative to $A$ such that $Z$ is built from $A$ by attaching cells of dim $> n$.

Proof: take $(Z, A)$ to be an $n$-connected model of $(X, A)$. We claim: $Z \xrightarrow{h.e.} X$ relative $A$. In fact, there exists $f : Z \rightarrow X$ such that $f_*$ is an isomorphism on $\pi_i$ when $i > n$ and is injective on $\pi_n$. For $i < n$, by the $n$-connectedness of the given model, $\pi_i(X) \cong \pi_i(A) \cong \pi_i(Z)$ where the isomorphisms are induced by $f$ since the following diagram commutes,

$$
\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
\uparrow & & \uparrow \\
A & \xrightarrow{id} & A
\end{array}
$$

(where the maps $A \hookrightarrow Z$ and $A \hookrightarrow X$ are inclusion maps.) For $i = n$, by $n$-connectedness we have the surjective maps:

$$
\pi_n(A) \xrightarrow{onto} \pi_n(Z) \xrightarrow{inj} \pi_n(X)
$$

So the induced map $f_* : \pi_n(A) \rightarrow \pi_n(X)$ is surjective. This, coupled with the Whitehead’s Theorem allows us to conclude that $f : Z \rightarrow X$ is a homotopy equivalence.

We make $f$ stationary on $A$ in the following way: define $W_f := M_f/\{a\} \times I \sim pt, \forall a \in A$ and look at the map $h : Z \rightarrow X$ given by the composition of $Z \hookrightarrow W_f \rightarrow X$ where the map from $W_f \rightarrow X$ is a deformation retract.

Claim: $Z$ is a deformation retract of $W_f$, thus giving us that $h$ is a homotopy equivalence relative to $A$. Proof of claim: We have $\pi_i(W_f) \cong \pi_i(X)$ (since $W_f$ is a deformation retract of $X$) and $\pi_i(X) \cong \pi_i(Z)$ since $X$ is homotopy equivalent to $Z$. Using Whitehead’s theorem, we conclude that $Z$ is a deformation retract of $W_f$. 

Proof of the theorem on cellular approximation: Construct $Z$ as a union of subcomplexes $A = Z_n \subseteq Z_{n+1} \subseteq ...$ such that for each $k \geq n + 1$, $Z_k$ is obtained from $Z_{k-1}$ by attaching $k$-cells.

We will show by induction that we can construct $Z_k$ with a map $f_k : Z_k \rightarrow X$ such that $f_k|_A = id|_A$ and $f_k$ is injective on $\pi_i$ for $n \leq i < k$ and onto on $\pi_i$ for $n < i \leq k$. We start the induction at $k = n$, $Z_n = A$, in which case the conditions on $\pi_i$ are void.
Induction step: \( k \to k + 1 \). Consider \( \{ \phi_\alpha \}_\alpha \) where \( \phi_\alpha : S^k \to Z_k \) is the generator of \( \ker[f_k : \pi_k(Z_k) \to \pi_k(X)] \). Define \( Y_{k+1} := Z_k \cup \bigcup_{\alpha} e^{k+1}_\alpha \),

where \( e^{k+1}_\alpha \) is a \((k + 1)\) cell attached to \( Z_k \) along \( \phi_\alpha \).

Then \( f_k : Z_k \to X \) extends to \( Y_{k+1} \). Indeed, \( f \circ \phi_\alpha : S^k \to Z_k \to X \) is nullhomotopic, since \( [f \circ \phi_\alpha] = f_k[\phi_\alpha] = 0 \). We hence get a map \( g : Y_{k+1} \to X \). It’s easy to check that the map is injective on \( \pi_i \) for \( n \leq i \leq k \), and onto on \( \pi_k \). In fact, since we extend \( f_k \) on \((k + 1)\)-cells, we only need to check the effect on \( \pi_k \). The elements of \( \ker(g) \) on \( \pi_k \) are represented by nullhomotopic maps (by construction) \( S^k \to Z_k \subset Y_{k+1} \to X \). So \( g_* \) is one-to-one on \( \pi_k \). Moreover, it is onto on \( \pi_k \) since, by hypothesis, the composition \( \pi_k(Z_k) \to \pi_k(Y_{k+1}) \to \pi_k(X) \) is onto.

Let \( \{ \phi_\beta : S^{k+1} \to X \} \) be a generator of \( \pi_{k+1}(X, x_0) \) and let \( Z_k = Y_{k+1} \vee S^{k+1}_\beta \). We now get a map \( f_{k+1} : Z_{k+1} \to X \), by defining \( f_{k+1}|_{S^{k+1}_\beta} = \phi_\beta \). This implies that \( f_{k+1} \) induces an epimorphism on \( \pi_{k+1} \). The remaining conditions on homotopy groups are easy to check. \( \square \)