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## 1 Fibrations. Fiber Bundles

### 1.1 Stiefel Manifolds

**Definition 1.** Define

\[ V_n(\mathbb{R}^k) : = \{ n\text{-frames in } \mathbb{R}^k \} , \]

where an \( n \)-frame is defined to be a sequence \( \{ v_1, \ldots, v_n \} \) of \( n \) linearly independent vectors in \( \mathbb{R}^k \) which are pairwise orthonormal: \( \langle v_i, v_j \rangle = \delta_{ij} \).

**Example 1.**

\[ V_1(\mathbb{R}^1) = S^{k-1} \]

**Example 2.**

\[ V_n(\mathbb{R}^n) \cong O(n) \]
We assign \( V_n(\mathbb{R}^k) \) the subspace topology induced from

\[
V_n(\mathbb{R}^k) \subset S^{k-1} \times \cdots \times S^{k-1},
\]

where \( S^{k-1} \times \cdots \times S^{k-1} \) has the usual product topology.

**Exercise:** Show that \( V_n(\mathbb{R}^k) \) is a CW complex.

### 1.2 Grassmann Manifolds

**Definition 2.** Define

\[
G_n(\mathbb{R}^k) := \{n\text{-dimensional vector subspaces in } \mathbb{R}^k\}.
\]

**Example 3.**

\( G_1(\mathbb{R}^k) = \mathbb{RP}^{k-1} \)

There is a map \( p : V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k) \) given by sending \( \{v_1, \ldots, v_n\} \mapsto \text{span}\{v_1, \ldots, v_n\} \).

**Claim 1.** \( p \) is onto, so \( G_n(\mathbb{R}^k) \) has the quotient topology.

**Proof** Let \( V \in G_n(\mathbb{R}^k) \). Choose a basis and make it orthonormal by the Gram-Schmidt procedure.

**Claim 2.** \( p \) is a fiber bundle with fiber \( O(n) = V_n(\mathbb{R}^k) \).

**Proof** Let \( V \in G_n(\mathbb{R}^k) \) and choose an orthonormal frame on \( V \). By projection and Gram-Schmidt, we get orthonormal frames on all “nearby” (in some neighborhood \( U \) of \( V \)) vector subspaces \( V' \). This is a continuous process. The existence of such frames allows us to identify \( p^{-1}(U) \) with \( U \times V_n(\mathbb{R}^n) \), where \( V_n(\mathbb{R}^n) \) is now identified as the fiber.

**Claim 3.** The same method gives a fiber bundle for all triples \( m < n \leq k \).

\[
\begin{array}{c}
V_{n-m}(\mathbb{R}^{k-m}) \longrightarrow V_n(\mathbb{R}^k) \longrightarrow V_m(\mathbb{R}^k) \\
\{v_1, \ldots, v_n\} \longrightarrow \{v_1, \ldots, v_m\}
\end{array}
\]  

(Think of orthogonal complements for the fiber.)
Example 4. If \( k = n \) in the bundle (1), we get the fiber bundle

\[
O(n - m) \longrightarrow O(n) \longrightarrow V_m(\mathbb{R}^n).
\]

If, moreover, \( m = 1 \), we get the fiber bundle

\[
\begin{array}{ccc}
O(n - 1) & \longrightarrow & \{0\} \\
A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}
\end{array}
\begin{array}{c}
\longrightarrow \\
B & \mapsto & Bu
\end{array}
\longrightarrow S^{n-1}
\]  

(2)

where \( u \in S^{n-1} \) is some fixed unit vector. 

Example 5. If \( m = 1 \) in the bundle (1), we get the fiber bundle

\[
V_{n-1}(\mathbb{R}^{k-1}) \longrightarrow V_n(\mathbb{R}^k) \longrightarrow S^{k-1}.
\]

(3)

Claim 4. \( V_n(\mathbb{R}^k) \) is \((k - n - 1)\)-connected.

**Proof** Exercise. Use the long exact sequence for bundle (3) and induction. 

The long exact sequence of homotopy groups for the bundle (2) shows that \( \pi_i(O(n)) \) is independent of \( n \) for \( n \) large. We call this stable homotopy group \( \pi_i(O) \).

**Bott Periodicity:** \( \pi_i(O) \) is periodic in \( i \) with period 8.

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>8</th>
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<tbody>
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<td>( \mathbb{Z}/2 )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>( \mathbb{Z} )</td>
</tr>
</tbody>
</table>

(There is a nice proof of this in Milnor’s *Morse Theory*, Section 24.)

Definition 3.

\[
V_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} V_n(\mathbb{R}^k) \quad \quad G_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)
\]

The space \( G_n(\mathbb{R}^\infty) \) is called the **classifying space for real vector bundles**.

\( G_n(\mathbb{R}^\infty) \) carries a lot of topological information. We can get a “limit” fiber bundle:

\[
\begin{array}{ccc}
O(n) & \longrightarrow & V_n(\mathbb{R}^\infty) \\
\longrightarrow & & \longrightarrow
\end{array}
\begin{array}{c}
G_n(\mathbb{R}^\infty)
\end{array}
\]

(4)

Claim 5. \( V_n(\mathbb{R}^\infty) \) is contractible.
Proof By the bundle (3) for $k \to \infty$, $\pi_i(V_n(\mathbb{R}^\infty)) = 0$ for all $i$. Using the CW structure and Whitehead’s Theorem shows that $V_n(\mathbb{R}^\infty)$ is contractible.

Alternatively, we can define an explicit homotopy $h_t : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ by

$$h_t(x_1, x_2, \ldots) := (1 - t)(x_1, x_2, \ldots) + t(0, x_1, x_2, \ldots).$$

Then $h_t$ is linear with $\ker h_t = \{0\}$. So $h_t$ preserves independence of vectors. Composing with Gram-Schmidt, we get a deformation of $V_n(\mathbb{R}^\infty)$ into the subspace of $n$-frames with first coordinate zero. Repeat this procedure $n$ times to get a deformation of $V_n(\mathbb{R}^\infty)$ to the subspace of $n$-frames with first $n$ coordinates zero.

Let $\{e_1, \ldots, e_n\}$ be the standard $n$-frame in $\mathbb{R}^\infty$. For an $n$-frame $\{v_1, \ldots, v_n\}$ of vectors with first $n$ coordinates zero, define a homotopy $k_t : V_n(\mathbb{R}^\infty) \longrightarrow V_n(\mathbb{R}^\infty)$ by

$$k_t(\{v_1, \ldots, v_n\}) := [(1 - t)\{v_1, \ldots, v_n\} + t\{e_1, \ldots, e_n\}] \circ (\text{Gram-Schmidt}).$$

Then $k_t$ preserves linear independence and orthonormality by Gram-Schmidt.

Composing $h_t$ and $k_t$, any $n$-frame is moved continuously to $\{e_1, \ldots, e_n\}$, the standard $n$-frame.

Over $\mathbb{C}$, we get fiber bundles

$$U(n) \hookrightarrow V_n(\mathbb{C}^k) \xrightarrow{p} G_n(\mathbb{C}^k).$$

As $k \to \infty$, we get

$$U(n) \hookrightarrow V_n(\mathbb{C}^\infty) \xrightarrow{p} G_n(\mathbb{C}^\infty).$$

Also

$$U(n - 1) \hookrightarrow U(n) \longrightarrow \mathbb{S}^{2n-1}.$$ (7)

The long exact sequence of homotopy groups then shows that $\pi_i(U(n))$ is stable for large $n$. This stable group $\pi_i(U)$ repeats itself with period 2.

1.3 Maps on Fibrations and Fiber Bundles

Definition 4. Given two fibrations $p_i : E_i \longrightarrow B$, $i = 1, 2$, a map $f : E_1 \longrightarrow E_2$ is fiber-preserving (f.p.) if the diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
B & & B
\end{array}
$$
commutes. Such an \( f \) is called a **fiber homotopy equivalence (f.h.e.)** if \( f \) is both f.p. and a homotopy equivalence, i.e., there is a map \( g : E_2 \longrightarrow E_1 \) such that \( f \) and \( g \) are f.p. and \( f \circ g \) and \( g \circ f \) are homotopic to \( id \) by f.p. maps.

**Exercise:** If \( p : E \longrightarrow B \) is a fibration over a contractible space \( B \), then \( p \) is f.h.e. to the trivial fibration \( B \times F \longrightarrow B \).

**Remark 1.**
1. Fibers of a fiber bundle are homeomorphic (to, say, \( p^{-1}(b) \)).
2. Fibers of a fibration are homotopy equivalent.

### 1.4 Turning Maps into Fibrations

Given a map \( f : A \longrightarrow B \), we would like to factor \( f \) as a composition of a homotopy equivalence and a fibration.

Construct

\[
A \xleftarrow{\text{h.e.}} \xrightarrow{\text{fibration}} E_f \xrightarrow{p} B
\]

where

\[
E_f := \{ (a, \gamma) \mid a \in A, \gamma : [0, 1] \longrightarrow B, \gamma(0) = f(a) \}
\]

is endowed with the compact-open topology. Define \( i : A \longrightarrow E_f \) by sending \( a \mapsto (a, c_{f(a)}) \), where \( c_{f(a)} \) denotes the constant path at \( f(a) \). Define \( p : E_f \longrightarrow B \) by sending \( (a, \gamma) \mapsto \gamma(1) \). Then \( f = p \circ i \).

**Theorem 1.1** \( p : E_f \longrightarrow B \) is a fibration.

**Example 6 (\( P_B \) and \( \Omega B \)).**

Take \( A = \{ b \} \) and \( f \) the inclusion \( f : \{ b \} \hookrightarrow B \). Then

\[
E_f = \{ (b, \gamma) \mid \gamma : [0, 1] \longrightarrow B, \gamma(0) = b \} =: P_B,
\]

the **path-space** of \( B \). The fiber over \( b \) of \( p : P_B \longrightarrow B \) is

\[
\Omega B := \{ \gamma : [0, 1] \longrightarrow B \mid \gamma(0) = \gamma(1) = b \},
\]

the **loop-space** of \( B \). We get a fibration

\[
\Omega B \xleftarrow{\text{contractible}} P_B \longrightarrow B,
\]

whence the long exact sequence of homotopy groups gives

\[
\pi_i(B) \cong \pi_{i-1}(\Omega B),
\]

for all \( i \).

The above example suggests Hurewicz can be proved by induction on the degree of connectivity. (If \( B \) is \( n \)-connected, then \( \Omega B \) is \( (n - 1) \)-connected.) We’ll give the details later via spectral sequences.
2 Spectral Sequences

2.1 Definitions

For what we are concerned with, we’ll start with a fibration $F \hookrightarrow E \twoheadrightarrow B$. Then we can intuitively regard spectral sequences as machines which, for example, take $H_*(F)$ and $H_*(B)$ as input data and output information about $H_*(E)$.

**Definition 5.** A spectral sequence is given by a sequence of (co-)chain groups $\{E^r_{*,*}, d^r\}_{r \geq 0}$ such that

$$E^0_{*,*} = H_*(E).$$

In more detail, we have groups (rings, modules, etc.) $\{E^r_{p,q}\}$ and maps

$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$

such that $(d^r)^2 = 0$ and

$$E^r_{p,q} := \frac{\ker (d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1})}{\text{im} (d^r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q})}.$$  

We focus on the first quadrant spectral sequence in $(p, q)$-space, so $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Hence, for any fixed $(p, q)$ in the first quadrant and for sufficiently large $r$, $d^r = 0$, so that $E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}$.

**Definition 6.** If $\{H_n\}_n$ are groups (rings, modules, etc.), we say the spectral sequence converges, or abuts, to $H_*$ if for each $n$ there is a filtration

$$H_n = D_{n,0} \supseteq D_{n-1,1} \supseteq \cdots \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

so that, for all $p, q$,

$$E^\infty_{p,q} = \frac{D_{p,q}}{D_{p-1,q+1}}.$$
To read off $H_n$ from $E_\infty$, we need to solve some extension problems. But if $E_{s,s}$ and $H_s$ are vector spaces, then

$$H_n = \bigoplus_{p+q=n} E_{p,q}^\infty.$$

**Remark 2.** The following observation is very useful in practice:

- If $E_{p,q}^\infty = 0$, for all $p + q = n$, then $H_n = 0$.
- If $H_n = 0$, then $E_{p,q}^\infty = 0$ for all $p + q = n$.

**Theorem 2.1 (Serre)** If $p : E \to B$ is a fibration with fiber $F$, and with $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence with $E_{p,q}^2 = H_p(B; H_q(F))$ converging to $H^*(E)$.

**Remark 3.** Fix some coefficient group $k$. Then:

- $E_{p,0}^2 = H_p(B; H_0(F; \mathbb{k})) = H_p(B; \mathbb{k})$, where $H_0(F; \mathbb{k}) = \mathbb{k}$
- $E_{0,q}^2 = H_0(B; H_q(F; \mathbb{k})) = H_q(F; \mathbb{k})$
2.2 Hurewicz Theorem Redux

We can now give a new proof of the

**Hurewicz Theorem:** If $X$ is $(n-1)$-connected, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i \leq n-1$ and $\pi_n(X) \cong H_n(X)$.

**Proof Idea:** Use the Serre spectral sequence for the path fibration:

$$
\begin{array}{ccc}
\Omega X & \hookrightarrow & P_X \\
\approx \{ \ast \} & \longrightarrow & X.
\end{array}
$$

Then the statement for $\pi_n$ would follow by induction from the sequence of isomorphisms:

$$
\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X) \cong H_n(X).
$$

The first isomorphism follows from the long exact sequence on homotopy groups of fibrations and the fact that $P_X \cong \{ \ast \}$. The second isomorphism follows from the induction hypothesis. So the problem reduces to showing the last isomorphism $H_{n-1}(\Omega X) \cong H_n(X)$.

For $n = 2$, i.e., the beginning of induction, we clearly have $H_1(X) = 0$ since $X$ is simply-connected, and

$$
\pi_2(X) \cong \pi_1(\Omega X) \cong \frac{\pi_1(\Omega X)}{\text{abelianization}} \cong H_1(\Omega X) \cong H_2(X).
$$

So it remains to show the isomorphism $H_1(\Omega X) \cong H_2(X)$. 

---

**Diagram:**

```
\begin{tikzpicture}
\node (E2) at (0,1) {$E^2$};
\node (HsF) at (0,0) {$H_*(F)$};
\node (HsB) at (1,0) {$H_*(B)$};
\node (0) at (0,-1) {0};
\node (0) at (1,-1) {0};
\draw[->] (E2) -- (HsF);
\draw[dashed,->] (HsF) -- (HsB);
\draw[->] (HsB) -- (0);
\end{tikzpicture}
```
Consider the $E_2$-page of the Serre spectral sequence for the path fibration.

We need to show $d^2 : H_2(X) \longrightarrow H_1(\Omega X)$ is an isomorphism.

Since $\{E^2_{p,q}\} \Rightarrow H_*(PX)$ and $PX$ is contactible, we have that $E^\infty_{p,q} = 0$ for all $p, q > 0$. Hence, if $d^2 : H_2(X) \longrightarrow H_1(\Omega X)$ is not an isomorphism, then $E^3_{0,1} \neq 0$, $E^3_{2,0} = \ker d^2 \neq 0$. But the differentials $d^3$ and higher will not affect $E^3_{0,1}$ and $E^3_{2,0}$. So they live at $E^\infty$, contradicting that $E^\infty = 0$ at these spots.

Now assume the statement of the Hurewicz theorem holds for $n - 1$ and prove it for $n$. To show that $\pi_n(X) \cong H_n(X)$, it suffices to show $H_{n-1}(\Omega X) \cong H_n(X)$.

By induction $H_{q \leq n-2}(\Omega X) = 0$, so $E^2_{p,q} = 0$ for all $0 < q < n - 1$. Hence, the differentials $d^2, d^3 \cdots d^{n-1}$ acting on the entries on the $p$-axis for $p \leq n$, do not affect these entries. The entries $H_n(X)$ and $H_{n-1}(\Omega X)$ are affected only by the differential $d^n$. Also, higher differentials starting with $d_{n+1}$ do not affect these entries. But since the spectral sequence abuts to $H_*(PX)$ with $PX$ contactible, all entries on the $E^\infty$-
page (except at the origin) must vanish. This forces $H_i(X) = 0$ for $1 \leq i \leq n - 1$. Similarly, $d^n : H_n(X) \longrightarrow H_{n-1}(\Omega X)$ must be an isomorphism.