1. Use homotopy groups in order to show that there is no retraction \( \mathbb{R}P^n \to \mathbb{R}P^k \) if \( n > k > 0 \).

2. Show that an \( n \)-connected, \( n \)-dimensional CW complex is contractible.

3. (Extension Lemma) Given a CW pair \( (X, A) \) and a map \( f : A \to Y \) with \( Y \) path-connected, show that \( f \) can be extended to a map \( X \to Y \) if \( \pi_{n-1}(Y) = 0 \) for all \( n \) such that \( X \setminus A \) has cells of dimension \( n \).

4. Show that a CW complex retracts onto any contractible subcomplex. (Hint: Use the above extension lemma.)

5. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes \( X_1 \subset X_2 \subset \cdots \) such that each inclusion \( X_i \hookrightarrow X_{i+1} \) is nullhomotopic. Conclude that \( S^\infty \) is contractible, and more generally, this is true for the infinite suspension \( \Sigma^\infty(X) := \bigcup_{n \geq 0} \Sigma^n(X) \) of any CW complex \( X \).

6. Use cellular approximation to show that the \( n \)-skeletons of homotopy equivalent CW complexes without cells of dimension \( n + 1 \) are also homotopy equivalent.

7. Show that a closed simply-connected 3-manifold is homotopy equivalent to \( S^3 \). (Hint: Use Poincaré Duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes.)

8. Suppose \( X \) is a CW complex with \( \tilde{H}_i(X; \mathbb{Z}) = 0 \) for all \( i \geq 0 \). Show that the suspension of \( X \) is contractible.

9. Show that a map \( f : X \to Y \) of connected CW complexes is a homotopy equivalence if it induces an isomorphism on \( \pi_1 \) and if a lift \( \tilde{f} : \tilde{X} \to \tilde{Y} \) to the universal covers induces an isomorphism on homology.

10. Show that \( \pi_7(S^4) \) is non-trivial. [Hint: It contains a \( \mathbb{Z} \)-summand.]
11. Prove that the space $SO(3)$ of orthogonal $3 \times 3$ matrices with determinant 1 is homeomorphic to $\mathbb{R}P^3$.

12. (a) Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n - 1$ and $m = 2n - 1$.
(b) Show that if there were fiber bundles $S^{n-1} \to S^{2n-1} \to S^n$ for all $n$, then the groups $\pi_i(S^n)$ would be finitely generated free abelian groups computable by induction, and non-zero if $i \geq n \geq 2$. 