

1 4.7:

#2: To find $T_4\{\sin x\}$, we observe that the $n^{th}$ term of the Taylor series for a function $f(x)$ is $\frac{f^{(n)}(0)}{n!}x^n$. Observing that the first, second, third, and fourth derivatives of $\sin x$ are $\cos x$, $-\sin x$, $-\cos x$, and $\sin x$ respectively, and that $\cos 0 = 1$ while $\sin 0 = 0$, we have $T_4\{\sin x\} = x - \frac{x^3}{6}$.

The error for the $n^{th}$ degree Taylor polynomial for a function $f(x)$ is $|\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}|$ for some $\xi$ between 0 and $x$. In this case, $n = 4$, and the fifth derivative of $\sin x$ is $\cos x$; so we have the error $E = \frac{\cos \xi}{120}x^5$. But, for any $\xi$, $-1 \leq \cos \xi \leq 1$, so $E \leq |\frac{1}{120}x^5|$; and since $|x| < 1$, this is less than $\frac{1}{120}$. So the error is no larger than $\frac{1}{120}$.

#3: (a) By elementary differentiation, we have that

\begin{align*}
f'(x) &= \frac{1}{3}(8 + x)^{-2/3} \\
f''(x) &= -\frac{2}{9}(8 + x)^{-5/3} \\
f^{(3)}(x) &= \frac{10}{27}(8 + x)^{-8/3}
\end{align*}

For $x = 0$, $(8 + x)^{1/3} = 2$, so $f(0) = 2$, $f'(0) = \frac{1}{12}$ and $f''(0) = -\frac{1}{144}$. So $T_2 f(x) = 2 + \frac{1}{12}x - \frac{1}{288}x^2$.

The error $|g^{1/3} - T_2 f(1)|$ is given by $E = |\frac{f^{(3)}(\xi)}{3!}x^3|$ at $x = 1$, for some $\xi$ between zero and $x$. $f^{(3)}(\xi) = \frac{10}{27}(8 + \xi)^{-8/3}$. Note that the exponent is negative; so this function is maximized when $\xi$ is minimized. When $\xi = 0$, $f^{(3)}(\xi) = \frac{5}{3436}$. So $|\frac{f^{(3)}(\xi)}{3!}x^3| \leq \frac{5}{3436}x^3 = \frac{5}{20736}$, since $x = 1$. So our error is at most $\frac{5}{20736}$, which is roughly 0.000025.

(b) We need one more derivative for this: $f^{(4)}(x) = -\frac{80}{81}(8 + x)^{-11/3}$. Now, from the first three derivatives, we know that

\begin{align*}
T_3 f(x) &= 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \frac{5}{20736}x^3
\end{align*}

Since $|f^{(4)}(\xi)|$ is largest when $\xi$ is 0, the error term is now

\begin{align*}
E &= \left|\frac{f^{(4)}(\xi)}{4!}x^4\right| \leq \left|\frac{5}{10368} - \frac{x^4}{4!}\right|
\end{align*}

We have $x = 1$, so this means that our error is at most $\frac{5}{10368}$, which is about 0.00002 - ten times better than our previous estimate.
#5: By simple application of our definition of a Taylor series, and taking the terms up to the eighth power of \(x\), we have that \(T_8 f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\). The error term is, as always, \(|f^{(9)}(\xi)\frac{x^9}{9!}|\), for \(\xi\) between 0 and \(x\); and \(f^{(9)}(\xi) = -\sin \xi\). So, since \(|-\sin \xi| \leq 1\), the error is at most \(|\frac{1}{9!}x^9|\). But we have \(|x| < 1\), so \(|\frac{1}{9!}x^9| \leq \frac{1}{9!}1^9 = \frac{1}{9!}\). For reference, this is about 0.0000003.

Next, we are asked to do the same thing with the ninth-degree polynomial. Note that the ninth derivative \(f\) at zero is \(-\sin(0) = 0\), so the coefficient of \(x^9\) needs to be zero - so again we have \(T_9 f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\). Now, however, the error term is

\[
\left|\frac{f^{(10)}(\xi)}{10!}\frac{x^{10}}{10!}\right| = \left|\frac{-\cos \xi}{10!}\frac{x^{10}}{10!}\right|
\]

\[
\leq \left|\frac{1}{10!}x^{10}\right|
\]

\[
\leq \frac{1}{10!}1^{10} = \frac{1}{10!}
\]

Which is roughly 0.0000003 - ten times better accuracy than before, without changing our Taylor approximation at all!

2 4.9:

#2: What it means to say that “\((1 + x^2)^2 - 1 = o(x^2)\)” is that \(\lim_{x \to 0} \frac{(1 + x^2)^2 - 1}{x^2} = 0\), simply by the definition of \(o(x^2)\). Note that \((1 + x^2)^2 - 1 = 1 + 2x^2 + x^4 - 1 = 2x^2 + x^4\). So

\[
\lim_{x \to 0} \frac{(1 + x^2)^2 - 1}{x^2} = \lim_{x \to 0} \frac{2x^2 + x^4}{x^2} = \lim_{x \to 0} (2 + x^2)
\]

\[
= 2
\]

The first two lines of this calculation are just algebra, with no limit calculations or anything - the only time we use a limit is in that last step.

But, as any elementary school student knows, \(2 \neq 0\). So \((1 + x^2)^2 - 1\) cannot be \(o(x^2)\), since it fails to satisfy the definition. So the statement is false.

#4: True. Let \(f\) and \(g\) be two arbitrary functions in \(o(x)\). Then \(\lim_{x \to 0} \frac{f(x) + g(x)}{x} = \lim_{x \to 0} \frac{f(x)}{x} + \frac{g(x)}{x}\). But, by the definition of \(o(x)\), \(\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{g(x)}{x} = 0\). So \(\lim_{x \to 0} \frac{f(x) + g(x)}{x} = 0 + 0 = 0\) - \(f(x) + g(x)\) is \(o(x)\), by the definition of \(o(x)\). So whenever two functions are \(o(x)\), their sum is also \(o(x)\); i.e., \(o(x) + o(x) = o(x)\).

#6: True. Let \(f\) and \(g\) be two arbitrary functions in \(o(x)\). Then \(\lim_{x \to 0} \frac{f(x)g(x)}{x} = \lim_{x \to 0} \frac{f(x)}{x} \cdot \frac{g(x)}{x}\). But, by the definition of \(o(x)\), \(\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{g(x)}{x} = 0\). So \(\lim_{x \to 0} \frac{f(x)}{x} \cdot \frac{g(x)}{x} = 0 \cdot 0 = 0\) - \(f(x) \cdot g(x)\) is \(o(x)\), by the definition of \(o(x)\). So whenever two functions are \(o(x)\), their product is also \(o(x)\); i.e., \(o(x) \cdot o(x) = o(x)\).

#8: False. Note that \(x^3\) is \(o(x^2)\), \(0\) is \(o(x^2)\), but \(x^3 - 0 = x^3 \neq o(x^3)\).
#10: True. Let f and g be two arbitrary functions, with f(x) = o(x) and g(x) = o(x^2). Then \( \lim_{x \to 0} f(x) + g(x) = \lim_{x \to 0} f(x) + \lim_{x \to 0} g(x) = f(0) + g(0) \). But, by the definition of o(x) and o(x^2), \( \lim_{x \to 0} f(x) = 0 \) and \( \lim_{x \to 0} g(x) = 0 \). So the sum of any o(x) function and any o(x^2) function is o(x); i.e., \( o(x) + o(x^2) = o(x) \).

#12: True. What we want to know is whether \( \lim_{x \to 0} \frac{1-\cos x}{x^2} = 0 \). Note that we know something very similar; we know that \( \lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \). Now, we do some manipulation:

\[
\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{(1-\cos x)/x^2}{x^2} = \lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{1}{x} \]

So the limit is indeed zero; by the definition of o(x), \( 1 - \cos x = o(x) \).

#14: (a) \( \sqrt{1+x^2} = 1 + o(x^k) \) if and only if \( \sqrt{1+x^2} - 1 = o(x^k) \). This is true if and only if \( \lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x^k} = 0 \).

\[
\lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x^k} = \lim_{x \to 0} \left( \frac{1+x^2}{x^k} - \frac{1}{x^k} \right) = \lim_{x \to 0} \left( \frac{1+x^2}{x^2} - \frac{1}{x^k} \right) = \lim_{x \to 0} \left( \frac{1}{x^2} + \frac{x^2}{x^2k} - \frac{1}{x^k} \right)
\]

Noting that \( \sqrt{1+x^2} = \frac{1}{x^k} \), this limit will be zero as long as \( \frac{x^2}{x^2k} \) goes to zero. But \( \frac{x^2}{x^2k} = x^{2-2k} \) only if \( 2-2k > 0 \), which is true only if \( k < 1 \). So \( \sqrt{1+x^2} = 1 + o(x^k) \) when \( k < 1 \).

(b) \( \sqrt{1+x^2} = 1 + o(x^k) \) if and only if \( \sqrt{1+x^2} - 1 = o(x^k) \). This is true if and only if \( \lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x^k} = 0 \).

\[
\lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x^k} = \lim_{x \to 0} \left( \frac{\sqrt{1+x^2}}{x^k} - \frac{1}{x^k} \right) = \lim_{x \to 0} \left( \frac{\sqrt{1+x^2}}{x^2k} - \frac{1}{x^k} \right) = \lim_{x \to 0} \left( \frac{1}{x^2k} + \frac{x^2}{x^2k} - \frac{1}{x^k} \right)
\]
Noting that $\sqrt[3]{\frac{x^2}{n^2}} = \frac{1}{n^2}$, this limit will be zero as long as $\frac{x^2}{n^2}$ goes to zero. But $\frac{x^2}{n^2} = x^2 - 3k$ only if $2 - 3k > 0$, which is true only if $k < 2/3$. So $\sqrt[3]{1 + x^2} = 1 + o(x^k)$ when $k < 2/3$.

(c) $1 - \cos(x^2) = o(x^k)$ only if $\lim_{x \to 0} \frac{1 - \cos(x^2)}{x^k} = 0$. Recall that $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$, so $\lim_{x \to 0} \frac{1 - \cos(x^2)}{x^2} = 0$. Therefore

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{x^k} = \lim_{x \to 0} \frac{(1 - \cos(x^2))/x^4}{x^k/x^4} = \lim_{x \to 0} \frac{(1 - \cos(x^2))/x^4}{x^4-k} = \frac{1}{2} \lim_{x \to 0} x^{4-k}$$

But this is zero only if $4 - k > 0$; in other words, $k < 4$. So $1 - \cos(x^2) = o(x^k)$ only if $k < 4$.

(d) $1 - \cos(x)^2 = (\sin x)^2$, which is $o(x^k)$ only if $\lim_{x \to 0} \frac{(\sin x)^2}{x^2} = 0$. Recall that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, so $\lim_{x \to 0} \frac{(\sin x)^2}{x^2} = 1^2 = 1$. Then

$$\lim_{x \to 0} \frac{(\sin x)^2}{x^k} = \lim_{x \to 0} \frac{(\sin x)^2/x^2}{x^{k-2}} = \lim_{x \to 0} \frac{(\sin x)^2}{x^2} x^{2-k} = \lim_{x \to 0} x^{2-k}$$

which is zero only if $2 - k > 0$; i.e., $k < 2$.

#16: (a) $h'(x) = \frac{2-x^2}{(2-3x+x^2)^2}$, so $h_0 = h(0) = 0$ and $h_1 = h'(0) = \frac{1}{2}$.

(b) Notice that $h(x) \cdot (2 - 3x + x^2) = x$; so for $n \geq 2$, the $n^{th}$ coefficient in $h(x) \cdot (2 - 3x + x^2)$ is zero. Let $h(x) = h_0 + h_1x + h_2x^2 + \cdots + h_n x^n + \cdots$. Then we can multiply out $h$ by $2 - 3x + x^2$;

\[ x = 2h_0 + 2h_1 x + 2h_2 x^2 + \cdots + 2h_n x^n + \cdots \tag{27} \]

\[ + -3h_0 x - 3h_1 x^2 + 3h_2 x^3 + \cdots + -3h_n x^{n+1} + \cdots \tag{28} \]

\[ + h_0 x^2 + h_1 x^3 + h_2 x^4 + \cdots + h_n x^{n+2} + \cdots \tag{29} \]

\[ = 2h_0 + (2h_1 - 3h_0) x + (2h_2 - 3h_1 + h_0) x^2 + \cdots + (2h_{n-2} - 3h_{n-1} + h_n) x^{n-2} + \cdots \tag{30} \]

So for $n \geq 2$, $2h_{n+2} - 3h_{n+1} + h_n = 0$. Solving for $h_n$, we obtain $h_n = 3h_{n+1} - 2h_{n+2}$.

(c) Applying the recursion relation we just found, we have that $h_2 = 3h_1 - 2h_0 = 3/2; h_3 = 3h_2 - 2h_1 = 7/2; h_4 = 3h_3 - 2h_2 = 15/2; h_5 = 3h_4 - 2h_3 = 31/2$. 

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(d) \(2 - 3x + x^2 = (x-1)(x-2)\), so let \(\frac{-x}{x-1+x^2} = \frac{A}{x-1} + \frac{B}{x+2}\). Applying the Heaviside trick, we obtain \(A = -1, B = 2\). So \(\frac{-x}{x-1+x^2} = \frac{1}{x-1} - \frac{1}{x+2}\). But \(\frac{1}{x-1} = \frac{1}{x-1} = 1 + x + x^2 + \cdots + x^n + \cdots\), and \(\frac{2}{x-1} = -\frac{2}{2-x} = -\frac{1}{1-x/2} = 1 + (x/2) + (x/2)^2 + \cdots + (x/2)^n + \cdots\). So the sum of the two is, adding each term independently, \(2 + (1 + 1/2)x + (1 + 1/4)x^2 + \cdots + (1 + 1/2^n)x^n + \cdots\). So \(h_n = 1 + \frac{1}{2^n}\) (which, incidentally, means that \(h^{(n)}(0) = n!(1 + \frac{1}{2^n})\).

#19:

\[e^{1+t} = e \cdot e^t = e \cdot (1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots + \frac{1}{n!}t^n + \cdots) = e + et + \frac{e}{2!}t^2 + \cdots + \frac{e}{n!}t^n + \cdots\]

And there’s our Taylor series.

#21: \(\frac{1+t}{1-t} = (1 + t)^{\frac{1}{1-t}}\). Using the Taylor series for \(\frac{1}{1-t}\), we get

\[1 + t + t^2 + \cdots + t^n + \cdots\]

#23: Start with the expansion for \(\sin x\) and divide by \(x\) - conveniently, every term in the series for \(\sin x\) is divisible by \(x\). The expansion for \(\sin x\) is \(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{n!} + \cdots\); so the expansion for \(f\) must be \(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots + (-1)^n\frac{x^{2n}}{n!} + \cdots\).

#25: Let’s start with the Taylor series for \(e^t\) and \(\frac{1}{1-t}\) and multiply them together. \(e^t = 1 + t + \cdots + \frac{1}{n!}t^n + o(t^n)\), and \(\frac{1}{1-t} = 1 + t + \cdots + t^n + o(t^n)\). So

\[
e^t \frac{1}{1-t} = (1 + t + \cdots + \frac{1}{n!}t^n + o(t^n))(1 + t + \cdots + t^n + o(t^n)) = 1(1 + t + \cdots + t^n + o(t^n)) + t(1 + t + \cdots + t^n + o(t^n)) + \cdots + t^n(1 + t + \cdots + t^n + o(t^n)) + o(t^n)
\]

\[
= 1 + (t + t) + (t^2 + t^2 + \frac{1}{2}t^2) + \cdots + o(t^n)
\]

Calculating out the first few terms here should convince you that the coefficients follow the pattern \(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots\) and so on. So the general term is \((\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{m!})t^n\).
#26: Let $\frac{1}{\sqrt{1-t}} = c_0 + c_1 t + \sum_{n=2}^{\infty} c_n t^n + o(t^n)$. Note that $\frac{1}{\sqrt{1-t}}^2 = \frac{1}{1-t}$, so when we square this polynomial we should obtain $1 + t + t^2 + \cdots + t^n + o(t^n)$. We therefore have $c_0(1 + c_1 t + \cdots + c_n t^n + o(t^n)) + c_1(t_0 + c_1 t + \cdots + c_n t^n + o(t^n)) + \cdots + c_n t^n(c_0 + c_1 t + \cdots + c_n t^n + o(t^n)) + o(t^n) = 1 + t + t^2 + \cdots + t^n + o(t^n)$. So $c_0 = 1, c_0 c_1 + c_0 = 1$, and so on. Then $c_0 = 1, c_1 = \frac{1}{2}$, and a few more steps should allow you to notice that $c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}$. So the general term is $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} t^n$.

#28: Note that the derivative of arcsin $t$ is $\frac{1}{\sqrt{1-t^2}}$, which is very much like the situation in 26, with a $t^2$ rather than $t$. So we compute the Taylor series for $\frac{1}{\sqrt{1-t^2}}$ by simply replacing every $t$ with a $t^2$, resulting in $1 + \frac{1}{2} t^2 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} t^{2n} + \cdots$. Then, we integrate, term-by-term, obtaining $t + \frac{1}{2} \cdot \frac{1}{2} t^3 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \frac{1}{2n+1} t^{2n+1} + \cdots$.

#30: We simply multiply the $T_n$ approximations of $e^{-t}$ and sin $2t$ together: we have $e^{-t} = 1 - t + \frac{1}{2} t^2 - \frac{1}{3!} t^3 + \frac{1}{4} t^4 + o(t^4)$ and sin $2t = 2t - \frac{8}{3} t^3 + o(t^4)$. Multiplying these out, we have $e^{-t} \sin 2t = (1 - t + \frac{1}{2} t^2 - \frac{1}{3!} t^3 + \frac{1}{4} t^4 + o(t^4))(2t - \frac{8}{3} t^3 + o(t^4)) = 2t - \frac{8}{3} t^3 + o(t^4)(2t - \frac{8}{3} t^3 + o(t^4)) = 2t - \frac{8}{3} t^3 + o(t^4)$, which you yourself can verify will evaluate to $2t - 2t^2 + \frac{1}{3} t^3 - \frac{7}{12} t^4 + \frac{5}{72} t^5 - \frac{1}{18} t^6 + o(t^6)$. But nothing above the $t^4$ term is trustworthy here, since the $o(t^4)$ contains $t^5$ terms and on up. So our $T_n = 2t - 2t^2 + \frac{7}{6} t^3 + t^4$.

#32: The integral of $\frac{1}{\sqrt{1+2t+t^2}}$ is $\frac{3}{4}(1 + 2t + t^2)^{4/3} = \frac{3}{4}(1 + 2t + t^2)^{\frac{4}{3}}$. So whatever the expansion for $\frac{1}{\sqrt{1+2t+t^2}}$ may be, when we integrate it we should get $\frac{3}{4}(1 + 2t + t^2)$ times what we started with. If our Taylor series is $c_0 + c_1 t + \cdots + c_n t^n + \cdots$, then its integral is $1 + c_0 t + \frac{c_1}{2} t^2 + \cdots + \frac{c_n}{n+1} t^{n+1} + \cdots$; and $\frac{3}{4}(1 + 2t + t^2)(c_0 + c_1 t + \cdots + c_n t^n + \cdots)$ is $\frac{3}{4} c_0 + \left(\frac{3}{4} c_1 + \frac{3}{2} c_0 \right) t + \cdots + \left(\frac{3}{4} c_n + \frac{3}{2} c_{n-1} + \frac{3}{2} c_{n-2} \right) t^n + \cdots$. These two must be equal, so their coefficients must be equal: so for each $n$, $\frac{3}{4} c_n + \frac{3}{2} c_{n-1} + \frac{3}{2} c_{n-2} = \frac{c_n}{n+1}$. Solving for $c_n$, we obtain the recurrence relation $c_n = \frac{n(n+1)^3 + 3n-2}{1-3n}$ (as long as $n \geq 2$, otherwise this doesn’t make sense). For $n = 0$ and $n = 1$, you can verify that $c_0 = 1$ and $c_1 = \frac{1}{2}$.

#34: Note that sin $t$ cos $t = \frac{1}{2}$ sin $2t$. Then, substituting $2t$ into the Taylor series for sin $2t$ and multiplying the whole thing by $\frac{1}{2}$, we have $t - \frac{1}{3} t^3 + \cdots + (-1)^n \frac{2^{2n+1}}{(2n+1)!} t^{2n+1} + \cdots$. 

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3 4.11

#1: (a) \( T_2(\sin t) = 0 + t + 0t^2 = t \). The remainder term \( R_2(\sin t) \) is \( \frac{|f^{(3)}(\xi)|}{3!} t^3 \) for \( \xi \) between 0 and \( t \). In this case, \( f^{(3)} \) is the third derivative of \( \sin t \), which is \(-\sin t\); this is bounded above and below by 1 and \(-1\) respectively, so we have \( R_2(\sin t) \leq \frac{1}{3!} t^3 \). But \( t \) is at most 0.5, so this is at most \( \frac{1}{3!} \frac{1}{4} = \frac{1}{24} \).

(b) We’ll approximate \( \int_0^{0.5} \sin x^2 dx \) using \( T_2(\sin x) \), where \( t = x^2 \). So our approximation will be \( \int_0^{0.5} x^2 dx = \frac{1}{24} \).

How far off are we? By the above calculation, the error in \( T_2 \) for any \( t \) less than 0.5 is at most \( \frac{1}{24} \). Here, \( t = x^2 \), and \( x^2 \) is at most 0.25, so this upper bound is still correct. So the worst case scenario is that we were always off by \( \frac{1}{24} \); our error is then \( \int_0^{0.5} \frac{1}{24} dx = \frac{1}{96} \).

#2: (a) \( 1 + t + \frac{1}{2} t^2 \).

(b) Let \( t = x^2 \); we are then approximating \( e^{x^2} \) with \( 1 + x^2 + \frac{1}{2} x^4 \). \( \int_0^{1} (1 + x^2 + \frac{1}{2} x^4)dx = \frac{41}{30} \).

The error term in the approximation for \( e^t \), with \( 0 \leq t \leq 1 \), is \( \frac{|e^t|}{2} \). Now, observe that \( e^t \) is an increasing function; larger \( t \) means larger \( e^t \). So \( e^t \) is maximized when \( \xi = t \). Then the upper bound is \( \frac{e^t}{2} t^3 \). But \( t \leq 1 \), so \( e^t \leq e \) and \( t^3 \leq 1 = 1 \); so the error term is bounded by \( \frac{e}{2} = \frac{e}{2} \). And to be even more casual about it, \( e \leq 3 \), so this is no more than \( \frac{3}{2} \).

So our approximation was off by at most \( \frac{1}{2} \) for each \( x \); so our integral was off by at most \( \int_0^{1} \frac{1}{2} dx = \frac{1}{2} \).

(c) The error term for the \( T_5 \) polynomial for \( e^t \) is \( \frac{|e^t|}{60} t^6 \). Again, \( e^t \leq e \) and \( t^6 \leq 1^6 = 1 \), so this is at most \( \frac{60}{1} = \frac{1}{240} \). Our error is therefore at most \( \int_0^{1} \frac{1}{240} dx = \frac{1}{240} \).

#5: We could work out the Taylor series for \( \sqrt{1 + x^4} \), but that would be painstaking and slow. Let’s be lazy, and instead work out the first few terms of the Taylor series for \( \sqrt{1 + t} \). The first few derivatives of \( \sqrt{1 + t} \) are \( \frac{1}{2} (1 + t)^{-1/2} \), \( -\frac{1}{4} (1 + t)^{-3/2} \), \( \frac{3}{8} (1 + t)^{-5/2} \), and \( -\frac{15}{16} (1 + t)^{-7/2} \). The \( n \)th term error is \( \frac{|f^{(n)}(\xi)|}{n!} t^n \), so it would be good to look at the maximum values of each of these derivatives. Since each of these derivatives have a negative exponent of \( t \), they are largest when \( t = 0 \). So the derivatives are at most \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{3}{8} \), and \( \frac{15}{16} \) respectively. Taking \( t \) at most 0.5\(^3\), since we will be using \( t = x^4 \) and \( x \) is at most 0.5\(^2\), the error terms are therefore at most \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{3}{8} \), and \( \frac{15}{16} \) for \( n = 1, 2, 3, 4 \) respectively. This last option isn’t quite good enough; \( \int_0^{0.5} \frac{15}{16} dx = \frac{15}{32} \) is still larger than our 10\(^{-4}\) limit. But we can keep calculating out the error terms this way; eventually, you can verify that we will reach an error of less than 2\( \times 10^{-4} \) when \( n = 7 \). So we’re going to use a 6\(^{th}\) degree Taylor approximation, which is \( 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{5}{256} + \frac{7}{256} + \frac{21}{1024} \).