Problem 1  Divide the polynomial \( f(x) = x^5 - 5x^3 + 7x + 2 \) by the polynomial \( p(x) = x^2 + x + 3 \) to find the quotient and remainder. Write your answers in the indicated places below.

The standard division process gives

Quotient = \( x^3 - x^2 - 7x + 10 \)

Remainder = \( 18x - 28 \)

Problem 2  Without doing any division, find the remainder if the polynomial \( f(x) = x^{99} - x^{55} + 3 \) is divided by the polynomial \( p(x) = x - 1 \). Write your answer in the indicated place below.

The remainder when a polynomial \( f(x) \) is divided by \( (x - c) \) is \( f(c) \). In this case \( c = 1 \) and \( f(1) = 1^{99} - 1^{55} + 3 = 3 \).

Thus

Remainder = \( 3 \).

Problem 3  Solve for \( x \) the equation \( \log(x) = \log(x + 6) - \log(x + 3) + \log(2) \).

Using the rules for logarithms, we get

\[ \log(x) = \log(x + 6) - \log(x + 3) + \log(2) = \log \left( \frac{2(x + 6)}{x + 3} \right) \]

Since the logarithm function is one-to-one, this gives the equation

\[ x = \frac{2(x + 6)}{x + 3} \]

which simplifies to \( x^2 + x - 12 = 0 \). Factoring, this gives \( (x - 3)(x + 4) = 0 \). Thus there are two possible solutions: \( x = 3 \) and \( x = -4 \). However, you cannot plug \( x = -4 \) into the original equation since the domain of the function \( y = \log(x) \) is \( x > 0 \). Thus the answer is

\[ x = 3. \]

Problem 4  Solve for \( y \) the equation \( 4^{2y+3} = \frac{1}{8} \).

Since \( 4 = 2^2 \) and \( \frac{1}{8} = 2^{-3} \), we can rewrite the equation as \( (2^2)^{2y+3} = 2^{-3} \), or \( 2^{4y+6} = 2^{-3} \). Since the function \( f(t) = 2^t \) is one-to-one, it follows that \( 4y + 6 = -3 \). Thus the answer is

\[ y = -\frac{9}{4}. \]

Problem 5  Solve for \( x \) the equation \( 2^x - 8 \left( 2^{-x} \right) = 7 \).

Multiplying the given equation by \( 2^x \), we get the equation \( (2^x)^2 - 8 = 7 \left( 2^x \right) \). This gives the quadratic equation

\[ (2^x)^2 - 7 \left( 2^x \right) - 8 = 0, \]

or

\[ (2^x - 8) \left( 2^x + 1 \right) = 0. \]

Thus we have two possibilities. Either \( 2^x = 8 \), in which case \( x = 3 \), or \( 2^x = -1 \). This last equation has no solutions since \( 2^x \) is always positive. Thus the answer is

\[ x = 3. \]
Problem 6  Find the polynomial of degree 5 which has a root at \( x = 2 \) of multiplicity 3, a root at \( x = -1 \), a root \( a = -2 \), and which has y-intercept \(-4\). Leave your answer in factored form.

The polynomial must have the form \( P(x) = c(x - 2)^3(x + 1)(x + 2) \). In order that the y-intercept is \(-4\) we must have \(-4 = P(0) = c(-2)^3(1)(2) = -16c \). Thus \( c = \frac{1}{4} \), and the answer is

\[
P(x) = \frac{1}{4}(x - 2)^3(x + 1)(x + 2).
\]

Problem 7  Find the polynomial with real coefficients of degree 5 which has a root at \( x = 2 - i \), a root at \( x = 2 \), a root of multiplicity 2 at \( x = 0 \), and which has leading coefficient 3. Completely multiply out your answer.

Since the polynomial has real coefficients, if \( 2 - i \) is a root, so is \( 2 + i \). Since the leading coefficient is 3, the polynomial is \( P(x) = 3(x - (2 - i))(x - (2 + i))(x - 2)x^2 \). Multiplying this out gives

\[
P(x) = 3x^5 - 18x^4 + 39x^3 - 30x^2.
\]

Problem 8

(a) (8 points) What are the domains of the two functions \( f(x) = 3e^{-2x} \) and \( g(x) = \log_2(x + 2) \)? Write your answers in interval form.

The domain of \( f(x) \) is all real numbers \( x \), so the answer is \((-\infty, +\infty) \). Since \( \log_2(t) \) is defined only for \( t > 0 \), the domain of \( g(x) \) is all real numbers \( x \) such that \( x + 2 > 0 \). Thus the domain is \( x > -2 \), and the answer is \((-2, +\infty) \).

(b) On the grid below, sketch the graphs of the two functions

\[
y = f(x) = 3e^{-2x} \quad \text{and} \quad y = g(x) = \log_2(x + 2).
\]
(c) Using information from the graphs in part (b), what can you say about the number of solutions to the equation $3e^{-2x} = \log_2(x + 2)$? (Do not try to solve the equation!)

Since the function $f(x)$ is monotone decreasing and goes to zero as $x \to +\infty$ while the function $g(x)$ is monotone increasing and goes to $+\infty$ as $x \to +\infty$, the two curves must intersect in a single point. Thus there is a unique solution to the equation.

**Problem 9** (12 points) Consider the rational function

$$f(x) = \frac{(x - 1)^2}{(x + 1)(x - 2)}.$$

(a) What is the domain of $f(x)$?

The domain is the set of all real numbers except $x = -1$ and $x = 2$.

(b) What is the $y$-intercept of the graph $y = f(x)$?

The $y$-intercept is the value of $f(x)$ when $x = 0$, and so is $y = \frac{1}{2}$.

(c) What are the $x$-intercepts of the graph $y = f(x)$?

The $x$-intercepts are the points where $f(x) = 0$ and in this case is $x = 1$.

(d) What are the vertical asymptotes of the graph $y = f(x)$?

The vertical asymptotes are at the points where the denominator equals zero. Thus there are two vertical asymptotes: $x = -1$ and $x = 2$.

(e) Is there a horizontal asymptote for the graph $y = f(x)$? If so, what is it?

Since $f(x) = \frac{x^2 - 2x + 1}{x^2 - x - 2}$ and this approaches the value 1 as $x$ gets very large positive or negative, there is a horizontal asymptote at $y = 1$.

(f) (8 points) Sketch the graph of $y = f(x)$:

![Graph of f(x)](image)
Problem 10  Find the roots of the equation $x^4 - x^3 - x^2 = 0$.

We can factor the equation and write it as $x^2(x^2 - x - 1) = 0$. Thus either $x^2 = 0$ or $x^2 - x - 1 = 0$. We use the quadratic formula to solve the second of these, and we find that the roots of the equation are

$$\left\{ 0, 0, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2} \right\}.$$ 

Problem 11  Find the roots of the equation $2x^3 + x^2 - 3x + 1 = 0$.

We look for rational roots of the form $\frac{c}{d}$. Then $c$ must divide 1 and $d$ must divide 2. Hence the only possible rational roots are $\left\{ 1, -1, \frac{1}{2}, -\frac{1}{2} \right\}$. By plugging in, we find that $x = \frac{1}{2}$ is a root. Hence $(x - \frac{1}{2})$ is a factor. Dividing by this, we find that

$$2x^3 + x^2 - 3x + 1 = (x - \frac{1}{2})(2x^2 + 2x - 2).$$

Using the quadratic equation to solve $2x^2 + 2x - 2 = 0$, we find that the roots of $2x^3 + x^2 - 3x + 1 = 0$ are

$$\left\{ \frac{1}{2}, \frac{-1 + \sqrt{5}}{2}, \frac{-1 - \sqrt{5}}{2} \right\}.$$