Problem 1. Suppose that \( z \) is not an integer. Prove that
\[
\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2} = \frac{\pi^2}{(\sin(\pi z))^2}
\]
by integrating the function
\[
f(\zeta) = \frac{\pi \cot(\pi \zeta)}{(\zeta + z)^2}
\]
over the circle \( |\zeta| = N + \frac{1}{2} \) with \( N \) an integer, \( N \geq |z| \), adding the residues of \( f \) insider the circle, and then letting \( N \) tend to infinity.

Problem 2. Prove that if \( f \) is an entire function and if there are constants \( A, B > 0 \) so that for all \( R > 0 \) and some integer \( k \) we have
\[
\sup_{|z|=R} |f(z)| \leq AR^k + B,
\]
then \( f \) is a polynomial of degree at most \( k \).

Problem 3. Let \( \{w_1, \ldots, w_N\} \) be distinct complex numbers with \( |w_j| = 1 \) for all \( 1 \leq j \leq N \). Prove that there exists a complex number \( z \) with \( |z| = 1 \) so that
\[
\prod_{j=1}^{N} |z - w_j| \geq 1.
\]
Must there exist a complex number \( z_0 \) with \( |z_0| = 1 \) so that
\[
\prod_{j=1}^{N} |z_0 - w_j| = 1?
\]

Problem 4. Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disk, and let \( \mathbb{S} = \{ w \in \mathbb{C} : \Re m[w] < \frac{\pi}{2} \} \) be the horizontal strip centered at 0 with width \( \pi \).

(a) Find an explicit biholomorphic mapping from \( \mathbb{S} \) to \( \mathbb{D} \); i.e. find a holomorphic function \( \Phi : \mathbb{S} \to \mathbb{D} \) which is one-to-one and onto, and whose inverse is also holomorphic.

(b) Let \( f : \mathbb{S} \to \mathbb{S} \) be a holomorphic function with \( f(0) = 0 \). Show that
\[
\left| \frac{e^{f(z)} - 1}{e^{f(z)} + 1} \right| < \left| \frac{e^z - 1}{e^z + 1} \right|.
\]

(c) Let \( f : \mathbb{S} \to \mathbb{S} \) be a holomorphic function with \( f(0) = 0 \). What can you conclude about \( f \) if \( f'(0) = 1 \)?

Problem 5. Let \( f \) be a holomorphic function defined in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \{x_n\} \) be an infinite sequence of distinct real numbers such that \( |x_n| < \frac{1}{2} \) for every \( n \), and suppose that \( f(x_n) \) is real for every \( n \). Prove that \( f(z) = f(\zeta) \) for every \( z \in \mathbb{D} \).

Problem 6. Let \( f \) be a holomorphic function defined in the upper half plane \( \mathbb{U} = \{ z \in \mathbb{C} : \Re m[z] > 0 \} \), and suppose that \( \Re m[f(z)] > 0 \) for all \( z \in \mathbb{U} \). Suppose also that \( f(i) = i \). What can you conclude about \( f \)?

Problem 7. Prove that the function
\[
f(z) = -\frac{1}{2} \left( z + \frac{1}{z} \right)
\]
is a biholomorphic mapping of the upper half-disk \( \mathbb{D}^+ = \{ z = x + iy : |z| < 1, y > 0 \} \) to the upper half plane \( \mathbb{U} = \{ w \in \mathbb{C} : \Re m[w] > 0 \} \).

Problem 8. Let \( \{ f_n \} \) be a sequence of holomorphic functions defined in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), and suppose that the sequence \( \{ f_n \} \) converges uniformly on compact subsets of \( \mathbb{D} \) to a limit \( F \). Suppose that the equation \( f_n(z) = 0 \) has no solution in \( \mathbb{D} \) for \( n = 1, 2, \ldots \). Prove that either \( F(z) \equiv 0 \) for all \( z \in \mathbb{D} \) or that \( F(z) = 0 \) has no solution in \( \mathbb{D} \).