Solutions to Exercises, Section 3.5

1. About how many hours will it take for a sample of radon-222 to have only one-eighth as much radon-222 as the original sample?

   SOLUTION  The half-life of radon-222 is about 92 hours, as can be seen in the chart in this section. To reduce the number of radon-222 atoms to one-eighth the original number, we need 3 half-lives (because $2^3 = 8$). Thus it will take 276 hours (because $92 \times 3 = 276$) to have only one-eighth as much radon-222 as the original sample.
2. About how many minutes will it take for a sample of nitrogen-13 to have only one sixty-fourth as much nitrogen-13 as the original sample?

**SOLUTION** The half-life of nitrogen-13 is about 10 minutes, as can be seen in the chart in this section. To reduce the number of nitrogen-13 atoms to one sixty-fourth the original number, we need 6 half-lives (because $2^6 = 64$). Thus it will take 60 minutes (because $10 \times 6 = 60$) to have only one sixty-fourth as much nitrogen-13 as the original sample.
3. About how many years will it take for a sample of cesium-137 to have only two-thirds as much cesium-137 as the original sample?

**SOLUTION** The half-life of cesium-137 is about 30 years, as can be seen in the chart in this section. Thus if we start with $a$ atoms of cesium-137 at time 0, then after $t$ years there will be

$$a \cdot 2^{-t/30}$$

atoms left. We want this to equal $\frac{2}{3}a$. Thus we must solve the equation

$$a \cdot 2^{-t/30} = \frac{2}{3}a.$$

To solve this equation for $t$, divide both sides by $a$ and then take the logarithm of both sides, getting

$$-\frac{t}{30} \log 2 = \log \frac{2}{3}.$$

Now multiply both sides by $-1$, replace $-\log 2$ by $\log \frac{3}{2}$, and then solve for $t$, getting

$$t = 30 \log \frac{3}{2} \approx 17.5.$$

Thus two-thirds of the original sample will be left after approximately 17.5 years.
4. About how many years will it take for a sample of plutonium-239 to have only 1% as much plutonium-239 as the original sample?

**SOLUTION** The half-life of plutonium-239 is about 24,110 years, as can be seen in the chart in this section. Thus if we start with $a$ atoms of plutonium-239 at time 0, then after $t$ years there will be

$$a \cdot 2^{-t/24110}$$

atoms left. We want this to equal $\frac{1}{100}a$. Thus we must solve the equation

$$a \cdot 2^{-t/24110} = \frac{1}{100}a.$$

To solve this equation for $t$, divide both sides by $a$ and then take the logarithm of both sides, getting

$$-\frac{t}{24110} \log 2 = \log \frac{1}{100}.$$

Now multiply both sides by $-1$, replace $-\log \frac{1}{100}$ by $\log 100$, and then solve for $t$, getting

$$t = 24110 \frac{\log 100}{\log 2} = 24110 \frac{2}{\log 2} \approx 160183.$$

Thus 1% of the original sample will be left after approximately 160,183 years.
5. Suppose a radioactive isotope is such that one-fifth of the atoms in a sample decay after three years. Find the half-life of this isotope.

**Solution** Let \( h \) denote the half-life of this isotope, measured in years. If we start with a sample of \( a \) atoms of this isotope, then after 3 years there will be

\[
a \cdot 2^{-3/h}
\]

atoms left. We want this to equal \( \frac{4}{5}a \). Thus we must solve the equation

\[
a \cdot 2^{-3/h} = \frac{4}{5}a.
\]

To solve this equation for \( h \), divide both sides by \( a \) and then take the logarithm of both sides, getting

\[
-\frac{3}{h} \log 2 = \log \frac{4}{5}.
\]

Now multiply both sides by \(-1\), replace \(-\log \frac{4}{5}\) by \(\log \frac{5}{4}\), and then solve for \( h \), getting

\[
h = 3 \frac{\log \frac{2}{5}}{\log \frac{5}{4}} \approx 9.3.
\]

Thus the half-life of this isotope is approximately 9.3 years.
6. Suppose a radioactive isotope is such that five-sixths of the atoms in a sample decay after four days. Find the half-life of this isotope.

**SOLUTION** Let \( h \) denote the half-life of this isotope, measured in days. If we start with a sample of \( a \) atoms of this isotope, then after 4 days there will be

\[
a \cdot 2^{-4/h}
\]

atoms left. We want this to equal \( \frac{1}{6}a \). Thus we must solve the equation

\[
a \cdot 2^{-4/h} = \frac{1}{6}a.
\]

To solve this equation for \( h \), divide both sides by \( a \) and then take the logarithm of both sides, getting

\[
-\frac{4}{h} \log 2 = \log \frac{1}{6}.
\]

Now multiply both sides by \(-1\), replace \(- \log \frac{1}{6}\) by \(\log 6\), and then solve for \( h \), getting

\[
h = 4 \frac{\log 2}{\log 6} = 1.55.
\]

Thus the half-life of this isotope is approximately 1.55 days.
7. Suppose the ratio of carbon-14 to carbon-12 in a mummified cat is 64% of the corresponding ratio for living organisms. About how long ago did the cat die?

**SOLUTION** The half-life of carbon-14 is 5730 years. If we start with a sample of \( a \) atoms of carbon-14, then after \( t \) years there will be

\[
a \cdot 2^{-t/5730}
\]

atoms left. We want to find \( t \) such that this equals 0.64\( a \). Thus we must solve the equation

\[
a \cdot 2^{-t/5730} = 0.64a.
\]

To solve this equation for \( t \), divide both sides by \( a \) and then take the logarithm of both sides, getting

\[
-\frac{t}{5730} \log 2 = \log 0.64.
\]

Now solve for \( t \), getting

\[
t = -\frac{5730 \log 0.64}{\log 2} \approx 3689.
\]

Thus the cat died about 3689 years ago. Carbon-14 cannot be measured with extreme accuracy. Thus it is better to estimate that the cat died about 3700 years ago (because a number such as 3689 conveys more accuracy than will be present in such measurements).
8. Suppose the ratio of carbon-14 to carbon-12 in a fossilized wooden tool is 20% of the corresponding ratio for living organisms. About how old is the wooden tool?

**SOLUTION** The half-life of carbon-14 is 5730 years. If we start with a sample of \( a \) atoms of carbon-14, then after \( t \) years there will be

\[ a \cdot 2^{-t/5730} \]

atoms left. We want to find \( t \) such that this equals 0.2\( a \). Thus we must solve the equation

\[ a \cdot 2^{-t/5730} = 0.2a. \]

To solve this equation for \( t \), divide both sides by \( a \) and then take the logarithm of both sides, getting

\[ -\frac{t}{5730} \log 2 = \log 0.2. \]

Now solve for \( t \), getting

\[ t = -5730 \frac{\log 0.2}{\log 2} \approx 13305. \]

Thus the wooden tool is about 13,305 years old. Carbon-14 cannot be measured with extreme accuracy. Thus it is better to estimate that the wooden tool is about 13,000 years old (because a number such as 13,305 conveys more accuracy than will be present in such measurements).
9. How many more times intense is an earthquake with Richter magnitude 7 than an earthquake with Richter magnitude 5?

**SOLUTION** Here is an informal but accurate solution: Each increase of 1 in the Richter magnitude corresponds to an increase in the size of the seismic wave by a factor of 10. Thus an increase of 2 in the Richter magnitude corresponds to an increase in the size of the seismic wave by a factor of $10^2$. Hence an earthquake with Richter magnitude 7 is 100 times more intense than an earthquake with Richter magnitude 5.

Here is a more formal explanation using logarithms: Let $S_7$ denote the size of the seismic waves from an earthquake with Richter magnitude 7 and let $S_5$ denote the size of the seismic waves from an earthquake with Richter magnitude 5. Thus

$$7 = \log \frac{S_7}{S_0} \quad \text{and} \quad 5 = \log \frac{S_5}{S_0}.$$  

Subtracting the second equation from the first equation, we get

$$2 = \log \frac{S_7}{S_0} - \log \frac{S_5}{S_0} = \log \left( \frac{S_7}{S_0} / \frac{S_5}{S_0} \right) = \log \frac{S_7}{S_5}.$$  

Thus

$$\frac{S_7}{S_5} = 10^2 = 100.$$  

Hence an earthquake with Richter magnitude 7 is 100 times more intense than an earthquake with Richter magnitude 5.
10. How many more times intense is an earthquake with Richter magnitude 6 than an earthquake with Richter magnitude 3?

**SOLUTION** Here is an informal but accurate solution: Each increase of 1 in the Richter magnitude corresponds to an increase in the size of the seismic wave by a factor of 10. Thus an increase of 3 in the Richter magnitude corresponds to an increase in the size of the seismic wave by a factor of $10^3$. Hence an earthquake with Richter magnitude 6 is 1000 times more intense than an earthquake with Richter magnitude 3.

Here is a more formal explanation using logarithms: Let $S_6$ denote the size of the seismic waves from an earthquake with Richter magnitude 6 and let $S_3$ denote the size of the seismic waves from an earthquake with Richter magnitude 3. Thus

$$6 = \log \frac{S_6}{S_0} \quad \text{and} \quad 3 = \log \frac{S_3}{S_0}.$$ 

Subtracting the second equation from the first equation, we get

$$3 = \log \frac{S_6}{S_0} - \log \frac{S_3}{S_0} = \log \left( \frac{S_6}{S_0} / \frac{S_3}{S_0} \right) = \log \frac{S_6}{S_3}.$$ 

Thus

$$\frac{S_6}{S_3} = 10^3 = 1000.$$ 

Hence an earthquake with Richter magnitude 6 is 1000 times more intense than an earthquake with Richter magnitude 3.
11. The 1994 Northridge earthquake in Southern California, which killed several dozen people, had Richter magnitude 6.7. What would be the Richter magnitude of an earthquake that was 100 times more intense than the Northridge earthquake?

**SOLUTION** Each increase of 1 in the Richter magnitude corresponds to an increase in the intensity of the earthquake by a factor of 10. Hence an increase in intensity by a factor of 100 (which equals $10^2$) corresponds to an increase of 2 in the Richter magnitude. Thus an earthquake that was 100 times more intense than the Northridge earthquake would have Richter magnitude $6.7 + 2$, which equals 8.7.
12. The 1995 earthquake in Kobe (Japan), which killed over 6000 people, had Richter magnitude 7.2. What would be the Richter magnitude of an earthquake that was 1000 times less intense than the Kobe earthquake?

**SOLUTION** Each increase of 1 in the Richter magnitude corresponds to an increase in the intensity of the earthquake by a factor of 10. Hence a decrease in intensity by a factor of 1000 (which equals $10^3$) corresponds to a decrease of 3 in the Richter magnitude. Thus an earthquake that was 1000 times less intense than the Kobe earthquake would have Richter magnitude $7.2 - 3$, which equals 4.2.
13. The most intense recorded earthquake in the state of New York occurred in 1944; it had Richter magnitude 5.8. The most intense recorded earthquake in Minnesota occurred in 1975; it had Richter magnitude 5.0. Approximately how many times more intense was the 1944 earthquake in New York than the 1975 earthquake in Minnesota?

**Solution** Let $S_N$ denote the size of the seismic waves from the 1944 earthquake in New York and let $S_M$ denote the size of the seismic waves from the 1975 earthquake in Minnesota. Thus

$$5.8 = \log \frac{S_N}{S_0} \quad \text{and} \quad 5.0 = \log \frac{S_M}{S_0}.$$ 

Subtracting the second equation from the first equation, we get

$$0.8 = \log \frac{S_N}{S_0} - \log \frac{S_M}{S_0} = \log \left( \frac{S_N}{S_0} / \frac{S_M}{S_0} \right) = \log \frac{S_N}{S_M}.$$ 

Thus

$$\frac{S_N}{S_M} = 10^{0.8} \approx 6.3.$$ 

In other words, the 1944 earthquake in New York was approximately 6.3 times more intense than the 1975 earthquake in Minnesota.
14. The most intense recorded earthquake in Wyoming occurred in 1959; it had Richter magnitude 6.5. The most intense recorded earthquake in Illinois occurred in 1968; it had Richter magnitude 5.3. Approximately how many times more intense was the 1959 earthquake in Wyoming than the 1968 earthquake in Illinois?

**SOLUTION** Let $S_W$ denote the size of the seismic waves from the 1959 earthquake in Wyoming and let $S_I$ denote the size of the seismic waves from the 1968 earthquake in Illinois. Thus

$$6.5 = \log \frac{S_W}{S_0} \quad \text{and} \quad 5.3 = \log \frac{S_I}{S_0}.$$

Subtracting the second equation from the first equation, we get

$$1.2 = \log \frac{S_W}{S_0} - \log \frac{S_I}{S_0} = \log \left( \frac{S_W}{S_0} / \frac{S_I}{S_0} \right) = \log \frac{S_W}{S_I}.$$

Thus

$$\frac{S_W}{S_I} = 10^{1.2} \approx 15.8.$$

In other words, the 1959 earthquake in Wyoming was approximately 15.8 times more intense than the 1968 earthquake in Illinois.
15. The most intense recorded earthquake in Texas occurred in 1931; it had Richter magnitude 5.8. If an earthquake were to strike Texas next year that was three times more intense than the current record in Texas, what would its Richter magnitude be?

**SOLUTION** Let $S_T$ denote the size of the seismic waves from the 1931 earthquake in Texas. Thus

$$5.8 = \log \frac{S_T}{S_0}.$$ 

An earthquake three times more intense would have Richter magnitude

$$\log \frac{3S_T}{S_0} = \log 3 + \log \frac{S_T}{S_0} \approx 0.477 + 5.8 = 6.277.$$ 

Because of the difficulty of obtaining accurate measurements, Richter magnitudes are usually reported with only one digit after the decimal place. Rounding off, we would thus say that an earthquake in Texas that was three times more intense than the current record would have Richter magnitude 6.3.
16. The most intense recorded earthquake in Ohio occurred in 1937; it had Richter magnitude 5.4. If an earthquake were to strike Ohio next year that was 1.6 times more intense than the current record in Ohio, what would its Richter magnitude be?

**SOLUTION** Let \( S_B \) denote the size of the seismic waves from the 1937 earthquake in Ohio (here we are using the notation \( S_B \) because Ohio is the “Buckeye State”; using \( S_O \) would look too much like \( S_0 \)). Thus

\[
5.4 = \log \frac{S_B}{S_0}.
\]

An earthquake 1.6 times more intense would have Richter magnitude

\[
\log \frac{1.6S_B}{S_0} = \log 1.6 + \log \frac{S_B}{S_0} \approx 0.204 + 5.4
\]

\[
= 5.604.
\]

Because of the difficulty of obtaining accurate measurements, Richter magnitudes are usually reported with only one digit after the decimal place. Rounding off, we would thus say that an earthquake in Ohio that was 1.6 times more intense than the current record would have Richter magnitude 5.6.
17. Suppose you whisper at 20 decibels and normally speak at 60 decibels.

(a) What is the ratio of the sound intensity of your normal speech to the sound intensity of your whisper?

(b) How many times louder does your normal speech seem as compared to your whisper?

**SOLUTION**

(a) Each increase of 10 decibels corresponds to multiplying the sound intensity by a factor of 10. Going from a 20-decibel whisper to 60-decibel normal speech means that the sound intensity has been increased by a factor of 10 four times. Because $10^4 = 10,000$, this means that the ratio of the sound intensity of your normal speech to the sound intensity of your whisper is 10,000.

(b) Each increase of 10 decibels results in a doubling of loudness. Here we have an increase of 40 decibels, so we have had an increase of 10 decibels four times. Thus the perceived loudness has increased by a factor of $2^4$. Because $2^4 = 16$, this means that your normal conversation seems 16 times louder than your whisper.
18. Suppose your vacuum cleaner makes a noise of 80 decibels and you normally speak at 60 decibels.

(a) What is the ratio of the sound intensity of your vacuum cleaner to the sound intensity of your normal speech?

(b) How many times louder does your vacuum cleaner seem as compared to your normal speech?

**SOLUTION**

(a) Each increase of 10 decibels corresponds to multiplying the sound intensity by a factor of 10. Going from 60-decibel normal speech to an 80-decibel vacuum cleaner means that the sound intensity has been increased by a factor of 10 twice. Because $10^2 = 100$, this means that the ratio of the sound intensity of the vacuum cleaner to the sound intensity of normal speech is 100.

(b) Each increase of 10 decibels results in a doubling of loudness. Here we have an increase of 20 decibels, so we have had an increase of 10 decibels twice. Thus the perceived loudness has increased by a factor of $2^2$. Because $2^2 = 4$, this means that the vacuum cleaner seems four times louder than normal speech.
19. Suppose an airplane taking off makes a noise of 117 decibels and you normally speak at 63 decibels.

(a) What is the ratio of the sound intensity of the airplane to the sound intensity of your normal speech?

(b) How many times louder does the airplane seem than your normal speech?

**SOLUTION**

(a) Let $E_A$ denote the sound intensity of the airplane taking off and let $E_S$ denote the sound intensity of your normal speech. Thus

$$117 = 10 \log \frac{E_A}{E_0} \quad \text{and} \quad 63 = 10 \log \frac{E_S}{E_0}. \quad (1)$$

Subtracting the second equation from the first equation, we get

$$54 = 10 \log \frac{E_A}{E_0} - 10 \log \frac{E_S}{E_0}. \quad (2)$$

Thus

$$5.4 = \log \frac{E_A}{E_0} - \log \frac{E_S}{E_0} = \log \left( \frac{E_A}{E_0} / \frac{E_S}{E_0} \right) = \log \frac{E_A}{E_S}. \quad (3)$$

Thus

$$\frac{E_A}{E_S} = 10^{5.4} \approx 251,189. \quad (4)$$

In other words, the airplane taking off produces sound about 250 thousand times more intense than your normal speech.
(b) Each increase of 10 decibels results in a doubling of loudness. Here we have an increase of 54 decibels, so we have had an increase of 10 decibels 5.4 times. Thus the perceived loudness has increased by a factor of $2^{5.4}$. Because $2^{5.4} \approx 42$, this means that the airplane seems about 42 times louder than your normal speech.
20. Suppose your cell phone rings at a noise of 74 decibels and you normally speak at 61 decibels.

(a) What is the ratio of the sound intensity of your cell phone ring to the sound intensity of your normal speech?

(b) How many times louder does your cell phone ring seem than your normal speech?

**SOLUTION**

(a) Let $E_C$ denote the sound intensity of your cell phone ring and let $E_S$ denote the sound intensity of your normal speech. Thus

\[ 74 = 10 \log \frac{E_C}{E_0} \quad \text{and} \quad 61 = 10 \log \frac{E_S}{E_0}. \]

Subtracting the second equation from the first equation, we get

\[ 13 = 10 \log \frac{E_C}{E_0} - 10 \log \frac{E_S}{E_0}. \]

Thus

\[ 1.3 = \log \frac{E_C}{E_0} - \log \frac{E_S}{E_0} = \log \left( \frac{E_C}{E_0} / \frac{E_S}{E_0} \right) = \log \frac{E_C}{E_S}. \]

Thus

\[ \frac{E_C}{E_S} = 10^{1.3} \approx 20. \]

In other words, your cell phone produces sound about 20 times more intense than your normal speech.
(b) Each increase of 10 decibels results in a doubling of loudness. Here we have an increase of 13 decibels, so we have had an increase of 10 decibels 1.3 times. Thus the perceived loudness has increased by a factor of $2^{1.3}$. Because $2^{1.3} \approx 2.5$, this means that your cell phone ring seems about 2.5 times louder than your normal speech.
21. Suppose a television is playing softly at a sound level of 50 decibels. What decibel level would make the television sound eight times as loud?

SOLUTION Each increase of ten decibels makes the television sound twice as loud. Because $8 = 2^3$, the sound level must double three times to make the television sound eight times as loud. Thus 30 decibels must be added to the sound level, raising it to 80 decibels.
Suppose a radio is playing loudly at a sound level of 80 decibels. What decibel level would make the radio sound one-fourth as loud?

**SOLUTION** Each increase of ten decibels makes the radio sound twice as loud. Because $4 = 2^2$, the sound level must halve twice to make the radio sound one-fourth as loud. Thus 20 decibels must be subtracted from the sound level, reducing it to 60 decibels.
23. Suppose a motorcycle produces a sound level of 90 decibels. What decibel level would make the motorcycle sound one-third as loud?

**SOLUTION** Each decrease of ten decibels makes the motorcycle sound half as loud. The sound level must be cut in half \( x \) times, where 
\[ \frac{1}{3} = \left( \frac{1}{2} \right)^x, \]
to make the motorcycle sound one-third as loud. This equation can be rewritten as 
\[ 2^x = 3. \]
Taking common logarithms of both sides gives 
\[ x \log 2 = \log 3, \]
which implies that 
\[ x = \frac{\log 3}{\log 2} \approx 1.585. \]

Thus the sound level must be decreased by ten decibels 1.585 times, meaning that the sound level must be reduced by 15.85 decibels. Because 90 \(-\) 15.85 = 74.15, a sound level of 74.15 decibels would make the motorcycle sound one-third as loud.
24. Suppose a rock band is playing loudly at a sound level of 100 decibels. What decibel level would make the band sound three-fifths as loud?

**Solution** Each decrease of ten decibels makes the motorcycle sound half as loud. The sound level must be cut in half \( x \) times, where 
\[
\frac{3}{5} = \left(\frac{1}{2}\right)^x,
\]
to make the rock band sound three-fifths as loud. This equation can be rewritten as 
\[
2^x = \frac{5}{3}.
\]
Taking common logarithms of both sides gives 
\[
x \log 2 = \log \frac{5}{3},
\]
which implies that 
\[
x = \frac{\log \frac{5}{3}}{\log 2} \approx 0.737.
\]
Thus the sound level must be decreased by ten decibels 0.737 times, meaning that the sound level must be reduced by 7.37 decibels. Because 100 \(-\) 7.37 = 92.63, a sound level of 92.63 decibels would make the rock band sound three-fifths as loud.
25. How many times brighter is a star with apparent magnitude 2 than a star with apparent magnitude 17?

**SOLUTION** Every five magnitudes correspond to a change in brightness by a factor of 100. Thus a change in 15 magnitudes corresponds to a change in brightness by a factor of $100^3$ (because $15 = 5 \times 3$). Because $100^3 = (10^2)^3 = 10^6$, a star with apparent magnitude 2 is one million times brighter than a star with apparent magnitude 17.
26. How many times brighter is a star with apparent magnitude 3 than a star with apparent magnitude 23?

**Solution** Every five magnitudes correspond to a change in brightness by a factor of 100. Thus a change in 20 magnitudes corresponds to a change in brightness by a factor of $100^4$ (because $20 = 5 \times 4$). Because $100^4 = (10^2)^4 = 10^8$, a star with apparent magnitude 3 is one hundred million times brighter than a star with apparent magnitude 23.
27. Sirius, the brightest star that can be seen from Earth (not counting the sun), has an apparent magnitude of \(-1.4\). Vega, which was the North Star about 12,000 years ago (slight changes in Earth’s orbit lead to changing North Stars every several thousand years), has an apparent magnitude of 0.03. How many times brighter than Vega is Sirius?

**SOLUTION** Let $b_V$ denote the brightness of Vega and let $b_S$ denote the brightness of Sirius. Thus

$$0.03 = \frac{5}{2} \log \frac{b_0}{b_V} \quad \text{and} \quad -1.4 = \frac{5}{2} \log \frac{b_0}{b_S}.$$  

Subtracting the second equation from the first equation, we get

$$1.43 = \frac{5}{2} \log \frac{b_0}{b_V} - \frac{5}{2} \log \frac{b_0}{b_S}.$$  

Multiplying both sides by $\frac{2}{5}$, we get

$$0.572 = \log \left( \frac{b_0}{b_V} \right) - \log \left( \frac{b_0}{b_S} \right) = \log \left( \frac{b_0}{b_V} \frac{b_S}{b_0} \right) = \log \frac{b_S}{b_V}.$$  

Thus

$$\frac{b_S}{b_V} = 10^{0.572} \approx 3.7.$$  

Thus Sirius is approximately 3.7 times brighter than Vega.
28. The full moon has an apparent magnitude of approximately $-12.6$. How many times brighter than Sirius is the full moon?

**SOLUTION** Let $b_S$ denote the brightness of Sirius and let $b_M$ denote the brightness of the full moon. Thus

$$-1.4 = \frac{5}{2} \log \frac{b_0}{b_S} \quad \text{and} \quad -12.6 = \frac{5}{2} \log \frac{b_0}{b_M}.$$  

Subtracting the second equation from the first equation, we get

$$11.2 = \frac{5}{2} \log \frac{b_0}{b_S} - \frac{5}{2} \log \frac{b_0}{b_M}.$$  

Multiplying both sides by $\frac{2}{5}$, we get

$$4.48 = \log \frac{b_0}{b_S} - \log \frac{b_0}{b_M} = \log \left( \frac{b_0}{b_S} / \frac{b_0}{b_M} \right) = \log \frac{b_M}{b_S}.$$  

Thus

$$\frac{b_M}{b_S} = 10^{4.48} \approx 30200.$$  

Thus the full moon is approximately 30,200 times brighter than Sirius.
Neptune has an apparent magnitude of about 7.8. What is the apparent magnitude of a star that is 20 times brighter than Neptune?

**SOLUTION** Each decrease of apparent magnitude by 1 corresponds to brightness increase by a factor of $100^{1/5}$. If we decrease the magnitude by $x$, then the brightness increases by a factor of $(100^{1/5})^x$. For this exercise, we want $20 = (100^{1/5})^x$. To solve this equation for $x$, take logarithms of both sides, getting

$$\log 20 = x \log 100^{1/5} = \frac{2x}{5}.$$  

Thus

$$x = \frac{5}{2} \log 20 \approx 3.25.$$  

Because $7.8 - 3.25 = 4.55$, we conclude that a star 20 times brighter than Neptune has apparent magnitude approximately 4.55.
30. What is the apparent magnitude of a star that is eight times brighter than Neptune?

**Solution**  Each decrease of apparent magnitude by 1 corresponds to brightness increase by a factor of $100^{1/5}$. If we decrease the magnitude by $x$, then the brightness increases by a factor of $(100^{1/5})^x$. For this exercise, we want $8 = (100^{1/5})^x$. To solve this equation for $x$, take logarithms of both sides, getting

$$\log 8 = x \log 100^{1/5} = \frac{2x}{5}.$$

Thus

$$x = \frac{5}{2} \log 8 \approx 2.26.$$

Because $7.8 - 2.26 = 5.54$, we conclude that a star eight times brighter than Neptune has apparent magnitude approximately 5.54.
31. Suppose $f$ is a function with exponential decay. Explain why the function $g$ defined by $g(x) = \frac{1}{f(x)}$ is a function with exponential growth.

**SOLUTION** Because $f$ is a function with exponential decay, there exist positive constants $c$ and $k$ and a number $b > 1$ such that

$$f(x) = cb^{-kx}$$

for every number $x$. Thus

$$\frac{1}{f(x)} = \frac{1}{c}b^{kx}$$

for every number $x$, and thus $g$ is a function with exponential growth.
32. Show that an earthquake with Richter magnitude \( R \) has seismic waves of size \( S_010^R \), where \( S_0 \) is the size of the seismic waves of an earthquake with Richter magnitude 0.

**Solution** Let \( S \) denote the size of the seismic waves of an earthquake with Richter magnitude \( R \). The definition of the Richter magnitude states that

\[
R = \log \frac{S}{S_0}.
\]

The definition of the common logarithm now implies that

\[
\frac{S}{S_0} = 10^R.
\]

Thus

\[
S = S_010^R.
\]
33. Do a web search to find the most intense earthquake in the United States in the last calendar year and the most intense earthquake in Japan in the last calendar year. Approximately how many times more intense was the larger of these two earthquakes than the smaller of the two?

**SOLUTION** The most intense earthquake in the United States in 2006 was a 6.7 magnitude earthquake near Kalaoa, Hawaii on 15 October 2006 (see http://earthquake.usgs.gov/regional/states/historical.php).

The most intense earthquake in Japan in 2006 was a 6.3 magnitude earthquake near Kyushu on 11 June 2006 (see http://earthquake.usgs.gov/regional/world/historical_country.php).

Let \( S_U \) denote the size of the seismic waves from the largest earthquake in the United States in 2006 and let \( S_J \) denote the size of the seismic waves from the largest earthquake in Japan in 2006. Thus

\[
6.7 = \log \frac{S_U}{S_0} \quad \text{and} \quad 6.3 = \log \frac{S_J}{S_0},
\]

Subtracting the second equation from the first equation, we get

\[
0.4 = \log \left( \frac{S_U}{S_0} / \frac{S_J}{S_0} \right) = \log \frac{S_U}{S_J}.
\]

Thus

\[
\frac{S_U}{S_J} = 10^{0.4} \approx 2.5.
\]
In other words, the most intense earthquake in the United States in 2006 was approximately 2.5 times more intense than the most intense earthquake in the Japan in 2006.
34. Show that a sound with \( d \) decibels has intensity \( E_0 10^{d/10} \), where \( E_0 \) is the intensity of a sound with 0 decibels.

**SOLUTION** Let \( E \) denote the intensity of a sound with \( d \) decibels. The definition of decibels states that

\[
d = 10 \log \frac{E}{E_0},
\]

which implies that

\[
\frac{d}{10} = \log \frac{E}{E_0},
\]

The definition of the common logarithm now implies that

\[
\frac{E}{E_0} = 10^{d/10}.
\]

Thus

\[
E = E_0 10^{d/10}.
\]
35. Find at least three different web sites giving the apparent magnitude of Polaris (the North Star) accurate to at least two digits after the decimal point. If you find different values on different web sites (as the author did), then try to explain what could account for the discrepancy (and take this as a good lesson in the caution necessary when using the web as a source of scientific information).

**SOLUTION**  
Polaris has apparent magnitude 2.11 according to www.glyphweb.com/esky/stars/polaris.html.  
Polaris has apparent magnitude 2.02 according to www.astro.uiuc.edu/ kaler/sow/polaris.html.  
Polaris has apparent magnitude 1.97 according to http://www.redshiftnow.ca/starwatch/learnmore.aspx.  

These differences in the claimed apparent magnitude of Polaris could be due to small errors in various measurements that are used to compute the apparent magnitude. Or perhaps these measurements were made in different years and the brightness of Polaris is changing. Actually, the brightness of Polaris varies in a cycle that takes about four days. Some sources may be reporting the apparent magnitude when Polaris is in the brightest part of that cycle; other sources may be reporting the apparent magnitude when Polaris is in another part of its four-day cycle.  

At any rate, the different numbers reported for the apparent magnitude of Polaris show that all these sources should display some
caution—reporting the apparent magnitude of Polaris with two digits after the decimal point gives an incorrect impression of the accuracy.
36. Write a description of the logarithmic scale used for the pH scale, which measures acidity (this will probably require use of the library or the web).

**SOLUTION** The pH of a liquid is defined to be

\[-\log H^+ ,\]

where $H^+$ is the concentration of positively charged hydrogen atoms in the liquid. Pure water at room temperature has a pH of about 7, which is considered neutral in terms of being acidic or alkaline. A liquid with a pH of less than 7 is considered to be acidic; a liquid with a pH of more than 7 is considered to be alkaline.

Because pH is defined in terms of a logarithm, a change of 1 in pH corresponds to a change by a factor of 10 in acidity. For example, a liquid with a pH of 4 is 10 times more acidic than a liquid with a pH of 5.