PRACTICE PROBLEMS

PARK, BAE JUN

Natural logarithm2

Math114 Section 307 & 309

(1) Suppose you deposit $1000 in a bank account and interest is compounded 5 times per year at annual interest rate 5%. Find the balance 5 years later.

Annual Interest rate : \( r = 0.05 \)
Each time interest rate : \( \frac{0.05}{5} = 0.01 \)
Since we compound interest 5-times per year, the balance after 1 year is \( 1000 \left( 1 + \frac{1}{100} \right)^5 \)
After 5 years, the balance is \( 1000 \left( \left( 1 + \frac{1}{100} \right)^5 \right)^5 = 1000 \left( 1 + \frac{1}{100} \right)^{25} \)
\[ \therefore 1000 \left( 1 + \frac{1}{100} \right)^{25} \text{ dollars} \]

(2) Suppose you deposit $2000 in a bank account and interest is compounded 5 times per year at annual interest rate 7%. Find the balance 10 years later.

Annual Interest rate : \( r = 0.07 \)
Each time interest rate : \( \frac{0.07}{5} = \frac{7}{500} \)
Since we compound interest 5-times per year, the balance after 1 year is \( 2000 \left( 1 + \frac{7}{500} \right)^5 \)
After 10 years, the balance is \( 2000 \left( \left( 1 + \frac{7}{500} \right)^5 \right)^{10} = 2000 \left( 1 + \frac{7}{500} \right)^{50} \)
\[ \therefore 2000 \left( 1 + \frac{7}{500} \right)^{50} \text{ dollars} \]
(3) Suppose you deposit $3000 in a bank account and interest is compounded 4 times per year at annual interest rate $0.03$. Find the balance 7 years later.

Annual Interest rate : \( r = 0.03 \)

Each time interest rate : \( \frac{0.03}{4} = \frac{3}{400} \)

Since we compound interest 4-times per year, the balance after 1 year is \( 3000(1 + \frac{3}{400})^4 \)

After 7 years, the balance is \( 3000[(1 + \frac{3}{400})^4]^7 = 3000(1 + \frac{3}{400})^{28} \)

\[ \therefore 3000(1 + \frac{3}{400})^{28} \text{ dollars} \]

(4) Suppose you deposit $1000 in a bank account and interest is compounded continuously at annual interest rate $5\%$. Find the balance 5 years later.

Since we compound interest continuously, the balance after 1 year is \( 1000e^{\frac{5}{100}} \) and the balance after \( t \) years is \( 1000e^{\frac{5}{100}t} \)

After 5 years, the balance is \( 1000e^{\frac{5}{100}5} = 1000e^{\frac{1}{4}} = 1000e^{0.25} \)

\[ \therefore 1000e^{0.25} \text{ dollars.} \]
(5) Suppose you deposit $2000 in a bank account and interest is compounded continuously at annual interest rate 7%. Find the balance 10 years later.

Since we compound interest continuously, the balance after 1 year is $2000e^{0.07}$ and the balance after $t$ years is $2000e^{0.07t}$.

After 10 years, the balance is $2000e^{0.07\times10} = 2000e^{0.7}$

\[ \therefore 2000e^{0.7} \text{ dollars.} \]

(6) Suppose you deposit $3000 in a bank account and interest is compounded continuously at annual interest rate 0.03. Find the balance 7 years later.

Since we compound interest continuously, the balance after 1 year is $3000e^{0.03}$ and the balance after $t$ years is $3000e^{0.03t}$.

After 7 years, the balance is $3000e^{0.03\times7} = 3000e^{0.21} = 3000e^{0.21}$

\[ \therefore 3000e^{0.21} \text{ dollars.} \]
(7) How much would you need to deposit in a bank account paying 4% annual interest compounded continuously so that at the end of 20 years you would have $20,000?

Let \( P \) be the initial amount.

The balance 20 years later is \( Pe^{0.04 \times 20} = Pe^{0.8} = 20,000 \)

\[ \Rightarrow P = 20,000e^{-0.8} \]

\( \therefore 20,000e^{-0.8} \) dollars.

(8) Suppose a country’s population increases by a total of 3% over a two-year period. What is the continuous growth rate for this country?

Let \( P \) be the initial population and assume the continuous growth rate is \( r \) per year.

The population 2 years later is \( Pe^{2r} = P(1 + 0.03) = P \cdot 1.03 \)

\[ e^{2r} = 1.03 \]

\[ \Rightarrow 2r = \ln 1.03 \]

\[ \Rightarrow r = \frac{1}{2} \ln 1.03 \]

\( \therefore \frac{1}{2} \ln 1.03 \) (or 50 ln 1.03%) per year
(9) About how many years does it take for money to double when compounded continuously at 3% per year?

Let \( P \) be the initial amount.

The balance \( t \) years later is \( Pe^{0.03t} = 2P \)

\[ \Rightarrow e^{0.03t} = 2 \]

\[ \Rightarrow 0.03t = \ln 2 \]

\[ \Rightarrow t = \frac{100}{3}\ln 2 \]

\[ \therefore \frac{100}{3}\ln 2 \text{ years later}. \]

(10) A bacteria colony grows to five times its original size in 3 hours. Find its continuous growth rate.

Let \( P \) be the initial size of the colony and assume its continuous growth rate is \( r \) per hour.

After 3 hours, its size is \( Pe^{3r} = 5P \)

\[ e^{3r} = 5 \]

\[ \Rightarrow 3r = \ln 5 \]

\[ \Rightarrow r = \frac{1}{3}\ln 5 \]

\[ \therefore \frac{1}{3}\ln 5 \text{ (or } \frac{100}{3}\ln 5\% \text{) per hour}. \]
(11) How much would you need to deposit in a bank account paying 7% annual interest compounded continuously so that at the end of 10 years you would have $15,000?

Let \( P \) be the initial amount.

The balance 10 years later is \( Pe^{0.07 \cdot 10} = Pe^{0.7} = 15,000 \)

\[ \Rightarrow P = 15,000e^{-0.7} \]

\( \therefore 15,000e^{-0.7} \) dollars.

(12) Suppose a country’s population increases by a total of 10% over a three-year period. What is the continuous growth rate for this country?

Let \( P \) be the initial population and assume the continuous growth rate is \( r \) per year.

The population 3 years later is \( Pe^{3r} = P(1 + 0.1) = P \cdot 1.1 \)

\[ e^{3r} = 1.1 \]

\[ \Rightarrow 3r = \ln 1.1 \]

\[ \Rightarrow r = \frac{1}{3} \ln 1.1 \]

\( \therefore \frac{1}{3} \ln 1.1 \) (or \( \frac{100}{3} \) ln 1.1%) per year
(13) About how many years does it take for money to double when compounded continuously at 5% per year?

Let \( P \) be the initial amount.

The balance \( t \) years later is \( Pe^{0.05t} = 2P \)

\[ e^{0.05t} = 2 \]

\[ 0.05t = \ln 2 \]

\[ t = \frac{100}{5} \ln 2 = 20 \ln 2 \]

\( \therefore 20 \ln 2 \) years later.

(14) A bacteria colony grows to six times its original size in 5 days. Find its continuous growth rate.

Let \( P \) be the initial size of the colony and assume its continuous growth rate is \( r \) per day.

After 5 days, its size is \( Pe^{5r} = 6P \)

\[ e^{5r} = 6 \]

\[ 5r = \ln 6 \]

\[ r = \frac{1}{5} \ln 6 \]

\( \therefore \frac{1}{5} \ln 6 \) (or \( 20 \ln 6\%) \) per day.
(15) Suppose a colony of bacteria has doubled in 5 hours. What is the approximate continuous growth rate of this colony of bacteria?

\[ Pe^{5r} = 2P \Rightarrow e^{5r} = 2 \Rightarrow 5r = \ln 2 \Rightarrow r = \frac{\ln 2}{5} \]

\[ \therefore \text{the continuous growth rate is } \frac{\ln 2}{5} \text{ per hour. (or } \frac{100 \ln 2}{5} \approx \frac{70}{5} = 14\% \text{ per hour)} \]

(16) Suppose a colony of bacteria has doubled in 2 hours. What is the approximate continuous growth rate of this colony of bacteria?

\[ Pe^{2r} = 2P \Rightarrow e^{2r} = 2 \Rightarrow 2r = \ln 2 \Rightarrow r = \frac{\ln 2}{2} \]

\[ \therefore \text{the continuous growth rate is } \frac{\ln 2}{2} \text{ per hour. (or } \frac{100 \ln 2}{2} \approx \frac{70}{2} = 35\% \text{ per hour)} \]