Solutions to Exercises, Section 4.5

1. How much would an initial amount of $2000, compounded continuously at 6% annual interest, become after 25 years?

   **SOLUTION**  After 25 years, $2000 compounded continuously at 6% annual interest would grow to $2000e^{0.06 \times 25}$ dollars, which equals $2000e^{1.5}$ dollars, which is approximately $8963$. 
2. 📐 How much would an initial amount of $3000, compounded continuously at 7% annual interest, become after 15 years?

**SOLUTION** After 15 years, $3000 compounded continuously at 7% annual interest would grow to $3000e^{0.07\times 15}$ dollars, which equals $3000e^{1.05}$ dollars, which is approximately $8573.$
3. How much would you need to deposit in a bank account paying 4% annual interest compounded continuously so that at the end of 10 years you would have $10,000?

**SOLUTION** We need to find $P$ such that

$$10000 = Pe^{0.04 \cdot 10} = Pe^{0.4}.$$ 

Thus

$$P = \frac{10000}{e^{0.4}} \approx 6703.$$ 

In other words, the initial amount in the bank account should be $\frac{10000}{e^{0.4}}$ dollars, which is approximately $6703$. 
4. How much would you need to deposit in a bank account paying 5% annual interest compounded continuously so that at the end of 15 years you would have $20,000?

**SOLUTION** We need to find $P$ such that

$$20000 = Pe^{0.05 \times 15} = Pe^{0.75}.$$  

Thus

$$P = \frac{20000}{e^{0.75}} \approx 9447.$$  

In other words, the initial amount in the bank account should be $\frac{20000}{e^{0.75}}$ dollars, which is approximately $9447.$
5. Suppose a bank account that compounds interest continuously grows from $100 to $110 in two years. What annual interest rate is the bank paying?

**SOLUTION** Let \( r \) denote the annual interest rate paid by the bank. Then

\[
110 = 100e^{2r}.
\]

Dividing both sides of this equation by 100 gives \( 1.1 = e^{2r} \), which implies that \( 2r = \ln 1.1 \), which is equivalent to

\[
r = \frac{\ln 1.1}{2} \approx 0.0477.
\]

Thus the annual interest is approximately 4.77%.
6. Suppose a bank account that compounds interest continuously grows from $200 to $224 in three years. What annual interest rate is the bank paying?

**SOLUTION** Let $r$ denote the annual interest rate paid by the bank. Then

$$224 = 200e^{3r}.$$ 

Dividing both sides of this equation by 200 gives $1.12 = e^{3r}$, which implies that $3r = \ln 1.12$, which is equivalent to

$$r = \frac{\ln 1.12}{3} \approx 0.0378.$$ 

Thus the annual interest is approximately 3.78%. 

7. Suppose a colony of bacteria has a continuous growth rate of 15% per hour. By what percent will the colony have grown after eight hours?

SOLUTION A continuous growth rate of 15% per hour means that we should set $r = 0.15$. If the colony starts at size $P$ at time 0, then at time $t$ (measured in hours) its size will be $Pe^{0.15t}$.

Because $0.15 \times 8 = 1.2$, after eight hours the size of the colony will be $Pe^{1.2}$, which is an increase by a factor of $e^{1.2}$ over the initial size $P$. Because $e^{1.2} \approx 3.32$, this means that the colony will be about 332% of its original size after eight hours. Thus the colony will have grown by about 232% after eight hours.
8. Suppose a colony of bacteria has a continuous growth rate of 20% per hour. By what percent will the colony have grown after seven hours?

SOLUTION A continuous growth rate of 20% per hour means that we should set $r = 0.2$. If the colony starts at size $P$ at time 0, then at time $t$ (measured in hours) its size will be $Pe^{0.2t}$.

Because $0.2 \times 7 = 1.4$, after seven hours the size of the colony will be $Pe^{1.4}$, which is an increase by a factor of $e^{1.4}$ over the initial size $P$. Because $e^{1.4} \approx 4.055$, this means that the colony will be about 405.5% of its original size after seven hours. Thus the colony will have grown by about 305.5% after seven hours.
9. Suppose a country's population increases by a total of 3% over a two-year period. What is the continuous growth rate for this country?

**SOLUTION** A 3% increase means that we have 1.03 times as much as the initial amount. Thus \(1.03P = Pe^{2r}\), where \(P\) is the country's population at the beginning of the measurement period and \(r\) is the country's continuous growth rate. Thus \(e^{2r} = 1.03\), which means that \(2r = \ln 1.03\). Thus \(r = \frac{\ln 1.03}{2} \approx 0.0148\). Thus the country's continuous growth rate is approximately 1.48% per year.
10. Suppose a country’s population increases by a total of 6% over a three-year period. What is the continuous growth rate for this country?

**SOLUTION** A 6% increase means that we have 1.06 times as much as the initial amount. Thus $1.06P = Pe^{3r}$, where $P$ is the country’s population at the beginning of the measurement period and $r$ is the country’s continuous growth rate. Thus $e^{3r} = 1.06$, which means that $3r = \ln 1.06$. Thus $r = \frac{\ln 1.06}{3} \approx 0.0194$. Thus the country’s continuous growth rate is approximately 1.94% per year.
11. Suppose the amount of the world's computer hard disk storage increases by a total of 200% over a four-year period. What is the continuous growth rate for the amount of the world's hard disk storage?

**SOLUTION** A 200% increase means that we have three times as much as the initial amount. Thus \( 3P = Pe^{4r} \), where \( P \) is amount of the world's hard disk storage at the beginning of the measurement period and \( r \) is the continuous growth rate. Thus \( e^{4r} = 3 \), which means that \( 4r = \ln 3 \). Thus \( r = \frac{\ln 3}{4} \approx 0.275 \). Thus the continuous growth rate is approximately 27.5%.
12. Suppose the number of cell phones in the world increases by a total of 150% over a five-year period. What is the continuous growth rate for the number of cell phones in the world?

**SOLUTION** A 150% increase means that we have 2.5 times as much as the initial amount. Thus \(2.5P = Pe^{5r}\), where \(P\) is amount of the world’s hard disk storage at the beginning of the measurement period and \(r\) is the continuous growth rate. Thus \(e^{5r} = 2.5\), which means that \(5r = \ln 2.5\). Thus \(r = \frac{\ln 2.5}{5} \approx 0.183\). Thus the continuous growth rate is approximately 18.3%.
Suppose a colony of bacteria has a continuous growth rate of 30% per hour. If the colony contains 8000 cells now, how many did it contain five hours ago?

**SOLUTION** Let $P$ denote the number of cells at the initial time five hours ago. Thus we have $8000 = Pe^{0.3 \times 5}$, or $8000 = Pe^{1.5}$. Thus

$$P = 8000/e^{1.5} \approx 1785.$$
14. Suppose a colony of bacteria has a continuous growth rate of 40% per hour. If the colony contains 7500 cells now, how many did it contain three hours ago?

**SOLUTION** Let \( P \) denote the number of cells at the initial time three hours ago. Thus we have \( 7500 = Pe^{0.4 \times 3} \), or \( 7500 = Pe^{1.2} \). Thus

\[
P = \frac{7500}{e^{1.2}} \approx 2259.
\]
Suppose a colony of bacteria has a continuous growth rate of 35% per hour. How long does it take the colony to triple in size?

**Solution**  Let $P$ denote the initial size of the colony, and let $t$ denote the time that it takes the colony to triple in size. Then $3P = Pe^{0.35t}$, which means that $e^{0.35t} = 3$. Thus $0.35t = \ln 3$, which implies that $t = \frac{\ln 3}{0.35} \approx 3.14$. Thus the colony triples in size in approximately 3.14 hours.
16. Suppose a colony of bacteria has a continuous growth rate of 70% per hour. How long does it take the colony to quadruple in size?

**Solution** Let $P$ denote the initial size of the colony, and let $t$ denote the time that it takes the colony to quadruple in size. Then $4P = Pe^{0.7t}$, which means that $e^{0.7t} = 4$. Thus $0.7t = \ln 4$, which implies that $t = \frac{\ln 4}{0.7} \approx 1.98$. Thus the colony quadruples in size in approximately 1.98 hours.
17. About how many years does it take for money to double when compounded continuously at 2% per year?

**SOLUTION** At 2% per year compounded continuously, money will double in approximately $\frac{70}{2}$ years, which equals 35 years.
18. About how many years does it take for money to double when compounded continuously at 10% per year?

SOLUTION  At 10% per year compounded continuously, money will double in approximately \( \frac{70}{10} \) years, which equals 7 years.
19. About how many years does it take for $200 to become $800 when compounded continuously at 2% per year?

SOLUTION At 2% per year, money doubles in approximately 35 years. For $200 to become $800, it must double twice. Thus this will take about 70 years.
20. About how many years does it take for $300 to become $2,400 when compounded continuously at 5% per year?

**SOLUTION** At 5% per year, money doubles in approximately 14 years. For $300 to become $2,400, it must double three times. Thus this will take about 42 years.
21. How long does it take for money to triple when compounded continuously at 5% per year?

**Solution** To triple an initial amount $P$ in $t$ years at 5% annual interest compounded continuously, the following equation must hold:

$$Pe^{0.05t} = 3P.$$ 

Dividing both sides by $P$ and then taking the natural logarithm of both sides gives $0.05t = \ln 3$. Thus $t = \frac{\ln 3}{0.05}$. Thus it would take $\frac{\ln 3}{0.05}$ years, which is about 22 years.
22. How long does it take for money to increase by a factor of five when compounded continuously at 7% per year?

SOLUTION To increase an initial amount $P$ by a factor of five in $t$ years at 7% annual interest compounded continuously, the following equation must hold:

$$Pe^{0.07t} = 5P.$$ 

Dividing both sides by $P$ and then taking the natural logarithm of both sides gives $0.07t = \ln 5$. Thus $t = \frac{\ln 5}{0.07}$. Thus it would take $\frac{\ln 5}{0.07}$ years, which is about 23 years.
23. Find a formula for estimating how long money takes to triple at \( R \) percent annual interest rate compounded continuously.

**SOLUTION** To triple an initial amount \( P \) in \( t \) years at \( R \) percent annual interest compounded continuously, the following equation must hold:

\[ Pe^{\frac{Rt}{100}} = 3P. \]

Dividing both sides by \( P \) and then taking the natural logarithm of both sides gives \( Rt/100 = \ln 3 \). Thus \( t = \frac{100 \ln 3}{R} \). Because \( \ln 3 \approx 1.10 \), this shows that money triples in about \( \frac{110}{R} \) years.
24. Find a formula for estimating how long money takes to increase by a factor of ten at \( R \) percent annual interest compounded continuously.

**Solution** To increase an initial amount \( P \) by a factor of 10 in \( t \) years at \( R \) percent annual interest compounded continuously, the following equation must hold:

\[
P e^{\frac{Rt}{100}} = 10P.
\]

Dividing both sides by \( P \) and then taking the natural logarithm of both sides gives \( \frac{Rt}{100} = \ln 10 \). Thus \( t = \frac{100 \ln 10}{R} \). Because \( \ln 10 \approx 2.30 \), this shows that money increases by a factor of ten in about \( \frac{230}{R} \) years.
Suppose that one bank account pays 5% annual interest compounded once per year, and a second bank account pays 5% annual interest compounded continuously. If both bank accounts start with the same initial amount, how long will it take for the second bank account to contain twice the amount of the first bank account?

**Solution** Suppose both bank accounts start with $P$ dollars. After $t$ years, the first bank account will contain $P(1.05)^t$ dollars and the second bank account will contain $Pe^{0.05t}$ dollars. Thus we need to solve the equation

$$\frac{Pe^{0.05t}}{P(1.05)^t} = 2.$$ 

The initial amount $P$ drops out of this equation (as expected), and we can rewrite this equation as follows:

$$2 = \frac{e^{0.05t}}{1.05^t} = \frac{(e^{0.05})^t}{1.05^t} = \left(\frac{e^{0.05}}{1.05}\right)^t.$$ 

Taking the natural logarithm of the first and last terms above gives

$$\ln 2 = t \ln \frac{e^{0.05}}{1.05} = t (\ln e^{0.05} - \ln 1.05) = t (0.05 - \ln 1.05),$$

which we can then solve for $t$, getting

$$t = \frac{\ln 2}{0.05 - \ln 1.05}.$$
Using a calculator to evaluate the expression above, we see that $t$ is approximately 573 years.
26. Suppose that one bank account pays 3% annual interest compounded once per year, and a second bank account pays 4% annual interest compounded continuously. If both bank accounts start with the same initial amount, how long will it take for the second bank account to contain 50% more than the first bank account?

**Solution** Suppose both bank accounts start with $P$ dollars. After $t$ years, the first bank account will contain $P(1.03)^t$ dollars and the second bank account will contain $Pe^{0.04t}$ dollars. Thus we need to solve the equation

\[
\frac{Pe^{0.04t}}{P(1.03)^t} = 1.5.
\]

The initial amount $P$ drops out of this equation (as expected), and we can rewrite this equation as follows:

\[
1.5 = \frac{e^{0.04t}}{1.03^t} = \left(\frac{e^{0.04}}{1.03}\right)^t.
\]

Taking the natural logarithm of the first and last terms above gives

\[
\ln 1.5 = t \ln \frac{e^{0.04}}{1.03} = t (\ln e^{0.04} - \ln 1.03) = t (0.04 - \ln 1.03),
\]

which we can then solve for $t$, getting

\[
t = \frac{\ln 1.5}{0.04 - \ln 1.03}.
\]
Using a calculator to evaluate the expression above, we see that $t$ is approximately 39 years.
Suppose a colony of 100 bacteria cells has a continuous growth rate of 30% per hour. Suppose a second colony of 200 bacteria cells has a continuous growth rate of 20% per hour. How long does it take for the two colonies to have the same number of bacteria cells?

**SOLUTION**  After $t$ hours, the first colony contains $100e^{0.3t}$ bacteria cells and the second colony contains $200e^{0.2t}$ bacteria cells. Thus we need to solve the equation

$$100e^{0.3t} = 200e^{0.2t}.$$ 

Dividing both sides by 100 and then dividing both sides by $e^{0.2t}$ gives the equation

$$e^{0.1t} = 2.$$ 

Thus $0.1t = \ln 2$, which implies that

$$t = \frac{\ln 2}{0.1} \approx 6.93.$$ 

Thus the two colonies have the same number of bacteria cells in a bit less than 7 hours.
28. Suppose a colony of 50 bacteria cells has a continuous growth rate of 35% per hour. Suppose a second colony of 300 bacteria cells has a continuous growth rate of 15% per hour. How long does it take for the two colonies to have the same number of bacteria cells?

**SOLUTION** After $t$ hours, the first colony contains $50e^{0.35t}$ bacteria cells and the second colony contains $300e^{0.15t}$ bacteria cells. Thus we need to solve the equation

$$50e^{0.35t} = 300e^{0.15t}.$$ 

Dividing both sides by 50 and then dividing both sides by $e^{0.15t}$ gives the equation

$$e^{0.2t} = 6.$$ 

Thus $0.2t = \ln 6$, which implies that

$$t = \frac{\ln 6}{0.2} \approx 8.96.$$ 

Thus the two colonies have the same number of bacteria cells in a bit less than 9 hours.
29. Suppose a colony of bacteria has doubled in five hours. What is the approximate continuous growth rate of this colony of bacteria?

**SOLUTION** The approximate formula for doubling the number of bacteria is the same as for doubling money. Thus if a colony of bacteria doubles in five hours, then it has a continuous growth rate of approximately 70/5% per hour. In other words, this colony of bacteria has a continuous growth rate of approximately 14% per hour.
30. Suppose a colony of bacteria has doubled in two hours. What is the approximate continuous growth rate of this colony of bacteria?

**SOLUTION** The approximate formula for doubling the number of bacteria is the same as for doubling money. Thus if a colony of bacteria doubles in two hours, then it has a continuous growth rate of approximately \((70/2)\%\) per hour. In other words, this colony of bacteria has a continuous growth rate of approximately 35% per hour.
31. Suppose a colony of bacteria has tripled in five hours. What is the continuous growth rate of this colony of bacteria?

**SOLUTION** Let $r$ denote the continuous growth rate of this colony of bacteria. If the colony initially contains $P$ bacteria cells, then after five hours it will contain $Pe^{5r}$ bacteria cells. Thus we need to solve the equation

$$Pe^{5r} = 3P.$$ 

Dividing both sides by $P$ gives the equation $e^{5r} = 3$, which implies that $5r = \ln 3$. Thus

$$r = \frac{\ln 3}{5} \approx 0.2197.$$ 

Thus the continuous growth rate of this colony of bacteria is approximately 22% per hour.
32. Suppose a colony of bacteria has tripled in two hours. What is the continuous growth rate of this colony of bacteria?

**SOLUTION** Let $r$ denote the continuous growth rate of this colony of bacteria. If the colony initially contains $P$ bacteria cells, then after two hours it will contain $Pe^{2r}$ bacteria cells. Thus we need to solve the equation

$$Pe^{2r} = 3P.$$ 

Dividing both sides by $P$ gives the equation $e^{2r} = 3$, which implies that $2r = \ln 3$. Thus

$$r = \frac{\ln 3}{2} \approx 0.5493.$$ 

Thus the continuous growth rate of this colony of bacteria is approximately 55% per hour.
Solutions to Problems, Section 4.5

33. Using compound interest, explain why

\[(1 + \frac{0.05}{n})^n < e^{0.05}\]

for every positive integer \(n\).

SOLUTION Suppose \(n\) is a positive integer. Consider two bank accounts, one paying 5% annual interest compounded \(n\) times per year, and one paying 5% annual interest compounded continuously. At the end of one year, the original amount in the first bank account has been multiplied by \((1 + \frac{0.05}{n})^n\) and the original amount in the second bank account has been multiplied by \(e^{0.05}\).

Continuous compounding produces more money than compounding \(n\) times per year because with continuous compounding there is more opportunity to earn interest on the interest. Thus

\[(1 + \frac{0.05}{n})^n < e^{0.05}.\]
34. Suppose that in Exercise 9 we had simply divided the 3% increase over two years by 2, getting 1.5% per year. Explain why this number is close to the more accurate answer of approximately 1.48% per year.

**SOLUTION** At low growth rates and for fairly short periods of time, the two methods of no compounding and continuous growth give approximately the same results because compounding does not play a huge factor in those circumstances. Because the growth was small (3%) and the time period was fairly short (2 years) in Exercise 9, the answer obtained by assuming no compounding is approximately the same as the more accurate answer using compounding.

We can also understand what is happening here by noting that at continuous growth rate $r$ for 2 years, the initial amount is multiplied by $e^{2r}$. If $r$ is small (say $r \approx 0.015$), then $2r$ is also small and thus

$$e^{2r} \approx 1 + 2r.$$

Note that $1 + 2r$ is the factor by which the initial amount would be multiplied if there was no compounding.
35. Suppose that in Exercise 11 we had simply divided the 200% increase over four years by 4, getting 50% per year. Explain why we should not be surprised that this number is not close to the more accurate answer of approximately 27.5% per year.

**SOLUTION**  In Exercise 11 the growth rate is high, and thus compounding makes a serious difference. Hence a method that ignores compounding does not give results that are even approximately correct.
In Section 3.4 we saw that if a population doubles every \( d \) time units, then the function \( p \) modeling this population growth is given by the formula

\[
p(t) = p_0 \cdot 2^{t/d},
\]

where \( p_0 \) is the population at time 0. Some books do not use the formula above but instead use the formula

\[
p(t) = p_0 e^{(t \ln 2)/d}.
\]

Show that the two formulas above are really the same.

[Which of the two formulas in this problem do you think is cleaner and easier to understand?]

**SOLUTION** We have

\[
e^{(t \ln 2)/d} = e^{\ln 2^{t/d}} = 2^{t/d}.
\]

Thus the two formulas are the same.
37. In Section 3.5 we saw that if a radioactive isotope has half-life \( h \), then the function modeling the number of atoms in a sample of this isotope is

\[
a(t) = a_0 \cdot 2^{-t/h},
\]

where \( a_0 \) is the number of atoms of the isotope in the sample at time 0. Many books do not use the formula above but instead use the formula

\[
a(t) = a_0 e^{-(t \ln 2)/h}.
\]

Show that the two formulas above are really the same.

[Which of the two formulas in this problem do you think is cleaner and easier to understand?]

**SOLUTION** We have

\[
e^{-(t \ln 2)/h} = e^{-\ln 2^{t/h}} = e^{\ln 2^{-t/h}} = 2^{-t/h}.
\]

Thus the two formulas are the same.
38. Explain why every function $f$ with exponential growth (see Section 3.4 for the definition) can be written in the form

$$f(x) = c e^{kx},$$

where $c$ and $k$ are positive constants.

**SOLUTION** Suppose $f$ is a function with exponential growth. Thus there are positive numbers $c$ and $u$ and a number $b > 1$ such that

$$f(x) = c b^{ux}$$

for every number $x$. Let $k = u \ln b$. Thus

$$k = u \ln b = \ln b^u,$$

which implies that $b^u = e^k$. Now

$$f(x) = c b^{ux} = c (b^u)^x = c (e^k)^x = c e^{kx}$$

for every number $x$. 