Solutions to Exercises, Section 5.2

In Exercises 1–8, convert each angle to radians.

1. \(15^\circ\)

   SOLUTION  
   Start with the equation
   
   \[360^\circ = 2\pi \text{ radians}.
   
   Divide both sides by 360 to obtain
   
   \[1^\circ = \frac{\pi}{180} \text{ radians}.
   
   Now multiply both sides by 15, obtaining
   
   \[15^\circ = \frac{15\pi}{180} \text{ radians} = \frac{\pi}{12} \text{ radians}.
   

2. $40^\circ$

   **SOLUTION** Start with the equation

   $$360^\circ = 2\pi \text{ radians}.$$ 

   Divide both sides by 360 to obtain

   $$1^\circ = \frac{\pi}{180} \text{ radians}.$$ 

   Now multiply both sides by 40, obtaining

   $$40^\circ = \frac{40\pi}{180} \text{ radians} = \frac{2\pi}{9} \text{ radians}.$$
3. $-45^\circ$

**SOLUTION** Start with the equation

$$360^\circ = 2\pi \text{ radians}.$$  

Divide both sides by 360 to obtain

$$1^\circ = \frac{\pi}{180} \text{ radians}.$$  

Now multiply both sides by $-45$, obtaining

$$-45^\circ = -\frac{45\pi}{180} \text{ radians} = -\frac{\pi}{4} \text{ radians}.$$
4. $-60^\circ$

**SOLUTION** Start with the equation

$$360^\circ = 2\pi \text{ radians.}$$

Divide both sides by 360 to obtain

$$1^\circ = \frac{\pi}{180} \text{ radians.}$$

Now multiply both sides by $-60$, obtaining

$$-60^\circ = -\frac{60\pi}{180} \text{ radians} = -\frac{\pi}{3} \text{ radians.}$$
5. 270°

**SOLUTION**  Start with the equation

\[ 360° = 2\pi \text{ radians}. \]

Divide both sides by 360 to obtain

\[ 1° = \frac{\pi}{180} \text{ radians}. \]

Now multiply both sides by 270, obtaining

\[ 270° = \frac{270\pi}{180} \text{ radians} = \frac{3\pi}{2} \text{ radians}. \]
6. 240°

**SOLUTION** Start with the equation

\[ 360° = 2\pi \text{ radians}. \]

Divide both sides by 360 to obtain

\[ 1° = \frac{\pi}{180} \text{ radians}. \]

Now multiply both sides by 240, obtaining

\[ 240° = \frac{240\pi}{180} \text{ radians} = \frac{4\pi}{3} \text{ radians}. \]
7. $1080^\circ$

**Solution**  Start with the equation

$$360^\circ = 2\pi \text{ radians}.$$  

Divide both sides by 360 to obtain

$$1^\circ = \frac{\pi}{180} \text{ radians}.$$  

Now multiply both sides by 1080, obtaining

$$1080^\circ = \frac{1080\pi}{180} \text{ radians} = 6\pi \text{ radians}.$$
8. 1440°

**SOLUTION** Start with the equation

\[ 360° = 2\pi \text{ radians}. \]

Divide both sides by 360 to obtain

\[ 1° = \frac{\pi}{180} \text{ radians}. \]

Now multiply both sides by 1440, obtaining

\[ 1440° = \frac{1440\pi}{180} \text{ radians} = 8\pi \text{ radians}. \]
9. $4\pi$ radians

**SOLUTION**  Start with the equation

\[ 2\pi \text{ radians} = 360^\circ. \]

Multiply both sides by 2, obtaining

\[ 4\pi \text{ radians} = 2 \cdot 360^\circ = 720^\circ. \]
10. $6\pi$ radians

**SOLUTION** Start with the equation

$$2\pi \text{ radians} = 360^\circ.$$  

Multiply both sides by 3, obtaining

$$6\pi \text{ radians} = 3 \cdot 360^\circ = 1080^\circ.$$
11. $\frac{\pi}{9}$ radians

**SOLUTION**  Start with the equation

$$2\pi \text{ radians} = 360^\circ.$$  

Divide both sides by 2 to obtain

$$\pi \text{ radians} = 180^\circ.$$  

Now divide both sides by 9, obtaining

$$\frac{\pi}{9} \text{ radians} = \frac{180^\circ}{9} = 20^\circ.$$
12. $\frac{\pi}{10}$ radians

**SOLUTION**  Start with the equation

\[ 2\pi \text{ radians} = 360^\circ. \]

Divide both sides by 2 to obtain

\[ \pi \text{ radians} = 180^\circ. \]

Now divide both sides by 10, obtaining

\[ \frac{\pi}{10} \text{ radians} = \frac{180^\circ}{10} = 18^\circ. \]
13. 3 radians

**SOLUTION** Start with the equation

\[
2\pi \text{ radians} = 360^\circ.
\]

Divide both sides by \(2\pi\) to obtain

\[
1 \text{ radian} = \frac{180^\circ}{\pi}.
\]

Now multiply both sides by 3, obtaining

\[
3 \text{ radians} = 3 \cdot \frac{180^\circ}{\pi} = \frac{540^\circ}{\pi}.
\]
14. 5 radians

**SOLUTION**  Start with the equation

\[ 2\pi \text{ radians} = 360^\circ. \]

Divide both sides by \(2\pi\) to obtain

\[ 1 \text{ radian} = \frac{180^\circ}{\pi}. \]

Now multiply both sides by 5, obtaining

\[ 5 \text{ radians} = 5 \cdot \frac{180^\circ}{\pi} = \frac{900^\circ}{\pi}. \]
15. $-\frac{2\pi}{3}$ radians

**SOLUTION** Start with the equation

$$2\pi \text{ radians} = 360^\circ.$$  

Divide both sides by 2 to obtain

$$\pi \text{ radians} = 180^\circ.$$  

Now multiply both sides by $-\frac{2}{3}$, obtaining

$$-\frac{2\pi}{3} \text{ radians} = -\frac{2}{3} \cdot 180^\circ = -120^\circ.$$
16. \(-\frac{3\pi}{4}\) radians

**SOLUTION** Start with the equation

\[ 2\pi \text{ radians} = 360^\circ. \]

Divide both sides by 2 to obtain

\[ \pi \text{ radians} = 180^\circ. \]

Now multiply both sides by \(-\frac{3}{4}\), obtaining

\[ -\frac{3\pi}{4} \text{ radians} = -\frac{3}{4} \cdot 180^\circ = -135^\circ. \]
17. Suppose an ant walks counterclockwise on the unit circle from the point \((0,1)\) to the endpoint of the radius that forms an angle of \(\frac{5\pi}{4}\) radians with the positive horizontal axis. How far has the ant walked?

**SOLUTION** The radius whose endpoint equals \((0,1)\) makes an angle of \(\frac{\pi}{2}\) radians with the positive horizontal axis. This radius corresponds to the smaller angle shown below.

Because \(\frac{5\pi}{4} = \pi + \frac{\pi}{4}\), the radius that forms an angle of \(\frac{5\pi}{4}\) radians with the positive horizontal axis lies \(\frac{\pi}{4}\) radians beyond the negative horizontal axis (half-way between the negative horizontal axis and the negative vertical axis). Thus the ant ends its walk at the endpoint of the radius corresponding to the larger angle shown below:

The ant walks along the thickened circular arc shown above. This circular arc corresponds to an angle of \(\frac{5\pi}{4} - \frac{\pi}{4}\) radians, which equals \(\frac{3\pi}{4}\) radians. Thus the distance walked by the ant is \(\frac{3\pi}{4}\).
18. Suppose an ant walks counterclockwise on the unit circle from the point \((-1, 0)\) to the endpoint of the radius that forms an angle of 6 radians with the positive horizontal axis. How far has the ant walked?

**SOLUTION** The radius whose endpoint equals \((-1, 0)\) makes an angle of \(\pi\) radians with the positive horizontal axis. This radius lies on the negative horizontal axis.

Because \(2\pi \approx 6.28\), the radius that forms an angle of 6 radians with the positive horizontal axis lies slightly below the positive horizontal axis, as shown below:

The ant walks along the thickened circular arc shown above. This circular arc corresponds to an angle of \(6 - \pi\) radians. Thus the distance walked by the ant is \(6 - \pi\).
19. Find the lengths of both circular arcs of the unit circle connecting the point \((1, 0)\) and the endpoint of the radius that makes an angle of 3 radians with the positive horizontal axis.

**SOLUTION** Because 3 is a bit less than \(\pi\), the radius that makes an angle of 3 radians with the positive horizontal axis lies a bit above the negative horizontal axis, as shown below. The thickened circular arc corresponds to an angle of 3 radians and thus has length 3. The entire unit circle has length \(2\pi\). Thus the length of the other circular arc is \(2\pi - 3\), which is approximately 3.28.
20. Find the lengths of both circular arcs of the unit circle connecting the point \((1, 0)\) and the endpoint of the radius that makes an angle of 4 radians with the positive horizontal axis.

SOLUTION  The radius that makes an angle of 4 radians with the positive horizontal axis is shown below. The thickened circular arc corresponds to an angle of 4 radians and thus has length 4. The entire unit circle has length \(2\pi\). Thus the length of the other circular arc is 
\[2\pi - 4\], which is approximately 2.28.
21. Find the lengths of both circular arcs of the unit circle connecting the point \((\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\) and the point whose radius makes an angle of 1 radian with the positive horizontal axis.

**SOLUTION** The radius of the unit circle whose endpoint equals \((\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\) makes an angle of \(\frac{\pi}{4}\) radians with the positive horizontal axis, as shown with the clockwise arrow below. The radius that makes an angle of 1 radian with the positive horizontal axis is shown with a counterclockwise arrow.

Thus the thickened circular arc above corresponds to an angle of \(1 + \frac{\pi}{4}\) and thus has length \(1 + \frac{\pi}{4}\), which is approximately 1.79. The entire unit circle has length \(2\pi\). Thus the length of the other circular arc below is \(2\pi - (1 + \frac{\pi}{4})\), which equals \(\frac{7\pi}{4} - 1\), which is approximately 4.50.
22. Find the lengths of both circular arcs of the unit circle connecting the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and the point whose radius makes an angle of 2 radians with the positive horizontal axis.

**SOLUTION** The radius of the unit circle whose endpoint equals $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ makes an angle of $\frac{3\pi}{4}$ radians with the positive horizontal axis, as shown with the longer arrow below. The radius that makes an angle of 2 radian with the positive horizontal axis is shown with the shorter arrow below.

Thus the thickened circular arc above corresponds to an angle of $\frac{3\pi}{4} - 2$ and thus has length $\frac{3\pi}{4} - 2$, which is approximately 0.356. The entire unit circle has length $2\pi$. Thus the length of the other circular arc below is $2\pi - \left(\frac{3\pi}{4} - 2\right)$, which equals $\frac{5\pi}{4} + 2$, which is approximately 5.927.
23. For a 16-inch pizza, find the area of a slice with angle $\frac{3}{4}$ radians.

SOLUTION  Pizzas are measured by their diameters; thus this pizza has a radius of 8 inches. Thus the area of the slice is $\frac{1}{2} \cdot \frac{3}{4} \cdot 8^2$, which equals 24 square inches.
24. For a 14-inch pizza, find the area of a slice with angle $\frac{4}{5}$ radians.

**SOLUTION** Pizzas are measured by their diameters; thus this pizza has a radius of 7 inches. Thus the area of the slice is $\frac{1}{2} \cdot \frac{2}{5} \cdot 7^2$, which equals $\frac{98}{5}$ square inches.
25. Suppose a slice of a 12-inch pizza has an area of 20 square inches. What is the angle of this slice?

**SOLUTION** This pizza has a radius of 6 inches. Let $\theta$ denote the angle of this slice, measured in radians. Then

$$20 = \frac{1}{2} \theta \cdot 6^2.$$ 

Solving this equation for $\theta$, we get $\theta = \frac{10}{9}$ radians.
26. Suppose a slice of a 10-inch pizza has an area of 15 square inches. What is the angle of this slice?

**SOLUTION** This pizza has a radius of 5 inches. Let $\theta$ denote the angle of this slice, measured in radians. Then

$$15 = \frac{1}{2} \theta \cdot 5^2.$$  

Solving this equation for $\theta$, we get $\theta = \frac{6}{5}$ radians.
27. Suppose a slice of pizza with an angle of \( \frac{5}{6} \) radians has an area of 21 square inches. What is the diameter of this pizza?

**SOLUTION** Let \( r \) denote the radius of this pizza. Thus

\[
21 = \frac{1}{2} \cdot \frac{5}{6} r^2.
\]

Solving this equation for \( r \), we get \( r = \sqrt{\frac{2 \times 21}{\frac{5}{6}}} \approx 7.1 \). Thus the diameter of the pizza is approximately 14.2 inches.
28. Suppose a slice of pizza with an angle of 1.1 radians has an area of 25 square inches. What is the diameter of this pizza?

**SOLUTION** Let \( r \) denote the radius of this pizza. Thus

\[
25 = \frac{1}{2} \times 1.1 r^2.
\]

Solving this equation for \( r \), we get

\[
r = \sqrt{\frac{25}{1.1}} \approx 6.74.
\]

Thus the diameter of the pizza is approximately 13.48 inches.
For each of the angles in Exercises 29–34, find the endpoint of the radius of the unit circle that makes the given angle with the positive horizontal axis.

29. \(\frac{5\pi}{6}\) radians

**SOLUTION**  For this exercise it may be easier to convert to degrees. Thus we translate \(\frac{5\pi}{6}\) radians to 150°.

The radius making a 150° angle with the positive horizontal axis is shown below. The angle from this radius to the negative horizontal axis equals 180° – 150°, which equals 30° as shown in the figure below. Drop a perpendicular line segment from the endpoint of the radius to the horizontal axis, forming a right triangle as shown below. We already know that one angle of this right triangle is 30°; thus the other angle must be 60°, as labeled below.

The side of the right triangle opposite the 30° angle has length \(\frac{1}{2}\); the side of the right triangle opposite the 60° angle has length \(\frac{\sqrt{3}}{2}\). Looking at the figure below, we see that the first coordinate of the endpoint of the radius is the negative of the length of the side opposite the 60° angle, and the second coordinate of the endpoint of the radius is the length of the side opposite the 30° angle. Thus the endpoint of the radius is \((-\frac{\sqrt{3}}{2}, \frac{1}{2})\).
30. $\frac{7\pi}{6}$ radians

**SOLUTION**  For this exercise it may be easier to convert to degrees. Thus we translate $\frac{7\pi}{6}$ radians to $210^\circ$.

The radius making a $210^\circ$ angle with the positive horizontal axis is shown below. The angle from the negative horizontal axis to this radius equals $210^\circ - 180^\circ$, which equals $30^\circ$ as shown in the figure below. Draw a perpendicular line segment from the endpoint of the radius to the horizontal axis, forming a right triangle as shown below. We already know that one angle of this right triangle is $30^\circ$; thus the other angle must be $60^\circ$, as labeled below.

The side of the right triangle opposite the $30^\circ$ angle has length $\frac{1}{2}$; the side of the right triangle opposite the $60^\circ$ angle has length $\frac{\sqrt{3}}{2}$. Looking at the figure below, we see that the first coordinate of the endpoint of the radius is the negative of the length of the side opposite the $60^\circ$ angle, and the second coordinate of the endpoint of the radius is the negative of the length of the side opposite the $30^\circ$ angle. Thus the endpoint of the radius is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. 
31. $-\frac{\pi}{4}$ radians

**SOLUTION** For this exercise it may be easier to convert to degrees. Thus we translate $-\frac{\pi}{4}$ radians to $-45^\circ$.

The radius making a $-45^\circ$ angle with the positive horizontal axis is shown below. Draw a perpendicular line segment from the endpoint of the radius to the horizontal axis, forming a right triangle as shown below. We already know that one angle of this right triangle is $45^\circ$; thus the other angle must also be $45^\circ$, as labeled below.

The hypotenuse of this right triangle is a radius of the unit circle and thus has length 1. The other two sides each have length $\frac{\sqrt{2}}{2}$. Looking at the figure below, we see that the first coordinate of the endpoint of the radius is $\frac{\sqrt{2}}{2}$ and the second coordinate of the endpoint of the radius is $-\frac{\sqrt{2}}{2}$. Thus the endpoint of the radius is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. 
32. $-\frac{3\pi}{4}$ radians

**SOLUTION** For this exercise it may be easier to convert to degrees. Thus we translate $-\frac{3\pi}{4}$ radians to $-135^\circ$.

The radius making a $-135^\circ$ angle with the positive horizontal axis is shown below. Draw a perpendicular line segment from the endpoint of the radius to the horizontal axis, forming a right triangle as shown below. Because $180 - 135 = 45$, one angle of this triangle equals $45^\circ$; thus the other angle must also be $45^\circ$, as labeled below.

The hypotenuse of this right triangle is a radius of the unit circle and thus has length 1. The other two sides each have length $\frac{\sqrt{2}}{2}$. Looking at the figure below, we see that the first coordinate of the endpoint of the radius is $-\frac{\sqrt{2}}{2}$ and the second coordinate of the endpoint of the radius is also $-\frac{\sqrt{2}}{2}$. Thus the endpoint of the radius is $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. 

![Diagram showing the right triangle and the coordinates of the radius endpoint.]
33. \( \frac{5\pi}{2} \) radians

**SOLUTION** Note that \( \frac{5\pi}{2} = 2\pi + \frac{\pi}{2} \). Thus the radius making an angle of \( \frac{5\pi}{2} \) radians with the positive horizontal axis is obtained by starting at the horizontal axis, making one complete counterclockwise rotation (which is \( 2\pi \) radians), and then continuing for another \( \frac{\pi}{2} \) radians. The resulting radius is shown below. Its endpoint is \((0, 1)\).
34. $\frac{11\pi}{2}$ radians

**SOLUTION** Note that $\frac{11\pi}{2} = 4\pi + \frac{3\pi}{2}$. Thus the radius making an angle of $\frac{11\pi}{2}$ radians with the positive horizontal axis is obtained by starting at the horizontal axis, making two complete counterclockwise rotations (which is $4\pi$ radians), and then continuing for another $\frac{3\pi}{2}$ radians. The resulting radius is shown below. Its endpoint is $(0, -1)$. 
Solutions to Problems, Section 5.2

For each of the angles listed in Problems 35–42, sketch the unit circle and the radius that makes the indicated angle with the positive horizontal axis. Be sure to include an arrow to show the direction in which the angle is measured from the positive horizontal axis.

35. \( \frac{5\pi}{18} \) radians

**SOLUTION**

An angle of \( \frac{5\pi}{18} \) radians, which equals 50°.
An angle of $\frac{1}{2}$ radian, which approximately equals $28.6^\circ$. 

SOLUTION
An angle of $2$ radians, which approximately equals $114.6^\circ$.  

SOLUTION
An angle of 5 radians, which approximately equals 286.5°.
Problem 39

39. $\frac{11\pi}{5}$ radians

SOLUTION

An angle of $\frac{11\pi}{5}$ radians, which equals $396^\circ$. 
40. $-\frac{\pi}{12}$ radians

**SOLUTION**

An angle of $-\frac{\pi}{12}$ radians, which equals $-15^\circ$. 
41. $-1$ radian

**SOLUTION**

An angle of $-1$ radian, which approximately equals $-57.3^\circ$. 
42. $-\frac{8\pi}{9}$ radians

**SOLUTION**

An angle of $-\frac{8\pi}{9}$ radians, which equals $-160^\circ$. 
43. Find the formula for the length of a circular arc corresponding to an angle of \( \theta \) radians on a circle of radius \( r \).

**SOLUTION** Suppose \( 0 < \theta \leq 2\pi \) and consider a circular arc on a unit circle of radius \( r \) corresponding to an angle of \( \theta \) radians, as shown in the thickened part of the unit circle below:

![Diagram of a circular arc](image)

The length of this circular arc equals the fraction of the entire circle taken up by this circular arc times the circumference of the entire circle. In other words, the length of this circular arc equals \( \frac{\theta}{2\pi} \) times \( 2\pi r \), which equals \( \theta r \).
44. Most dictionaries define acute angles and obtuse angles in terms of degrees. Restate these definitions in terms of radians.

**SOLUTION**  An angle between 0 radians and \( \frac{\pi}{2} \) radians is called an acute angle.

An angle between \( \frac{\pi}{2} \) radians and \( \pi \) radians is called an obtuse angle.
45. Find a formula (in terms of $\theta$) for the area of the region bounded by the thickened radii and the thickened circular arc shown below.

Here $0 < \theta < 2\pi$ and the radius shown above makes an angle of $\theta$ radians with the positive horizontal axis.

**SOLUTION**  The area of this region equals the fraction of the area inside the circle taken up by this region times the area inside the entire circle. In other words, the area of this region equals $\frac{\theta}{2\pi}$ times $\pi$, which equals $\frac{\theta}{2}$. 
46. Suppose the region bounded by the thickened radii and circular arc shown above is removed. Find a formula (in terms of $\theta$) for the perimeter of the remaining region inside the unit circle.

**SOLUTION**  The length of the thickened arc above is $\theta$, and thus the length of the remaining part of the circle equals $2\pi - \theta$. Each of the two radii has length 1. Thus the perimeter of the region in question is $2 + 2\pi - \theta$. 