GROUP ALGEBRAS WHOSE UNITS SATISFY A GROUP IDENTITY II

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Abstract. Let $K[G]$ be the group algebra of a torsion group $G$ over an infinite field $K$, and let $U = U(G)$ denote its group of units. A recent paper of A. Giambruno, S. K. Sehgal, and A. Valenti proved that if $U$ satisfies a group identity, then $K[G]$ satisfies a polynomial identity, thereby confirming a conjecture of Brian Hartley. Here we add a footnote to their result by showing that the commutator subgroup $G'$ of $G$ must have bounded period. Indeed, this additional fact enables us to obtain necessary and sufficient conditions for $U(G)$ to satisfy an identity.

§1. Introduction

Let $K[G]$ be the group algebra of a torsion group $G$ over an infinite field $K$, and let $U = U(G)$ denote its group of units. Then $U$ is said to satisfy a group identity if there exists a nontrivial word $w = w(x_1, \ldots, x_m)$ in the free group $\langle x_1, \ldots, x_m \rangle$ such that $w(u_1, \ldots, u_m) = 1$ for all $u_i \in U$. Recently, [GSV] confirmed a conjecture of Brian Hartley by showing that if $U$ satisfies a group identity, then $K[G]$ satisfies a polynomial identity. In particular, in view of [P, Corollaries 5.3.8 and 5.3.10], $G$ must have a large abelian section. But more can be said in this circumstance. For example, if char $K = 0$, then [GSV, Lemma 2.3] implies that $G$ is abelian. In other words, the commutator subgroup $G'$ has bounded period equal to 1, and $U(G)$ satisfies the identity $(x, y) = x^{-1}y^{-1}xy = 1$. We show here that a similar phenomenon occurs in characteristic $p > 0$. Note that Hartley’s conjecture for $p'$-groups in characteristic $p$ was verified in the earlier paper [GJV].

For the remainder of this paper, let $p$ be a fixed prime and let $K$ denote a fixed infinite field of characteristic $p$. Recall that a group $A$ is said to be $p$-abelian if its commutator subgroup $A'$ is a finite $p$-group, and that, by [P, Corollary 5.3.10], the group algebra $K[G]$ satisfies a polynomial identity if and only if $G$ has a normal $p$-abelian subgroup of finite index. This explains some of the group theoretic conditions which occur in part (ii) of our main result below.

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Theorem 1.1. Let $K[G]$ be the group algebra of a torsion group $G$ over an infinite field $K$ of characteristic $p > 0$. If $U = U(G)$ denotes the group of units of $K[G]$, then the following are equivalent.

i. $U$ satisfies a group identity.

ii. $G$ has a normal $p$-abelian subgroup of finite index, and $G'$ is a $p$-group of bounded period.

iii. $U$ satisfies $(x, y)^p = 1$ for some $k \geq 0$.

Of course, $(iii) \Rightarrow (i)$ is trivial. Furthermore, most of $(i) \Rightarrow (ii)$ is the main result of [GSV]. Thus, our goal here is to fine tune the latter paper to obtain precise necessary and sufficient conditions for $U$ to satisfy a group identity.

§2. The Implication $(i) \Rightarrow (ii)$

Here we assume that $G$ is torsion and that $U$ satisfies the group identity $w = 1$. Thus, in view of [GSV, Theorem] and [P, Corollary 5.3.10], we know that $G$ has a normal $p$-abelian subgroup of finite index. In particular, $G$ is locally finite, and [GSV, Lemma 2.3] implies that $G' = K[G]$ is a $p$-group.

Lemma 2.1. Suppose $U(G)$ satisfies the group identity $w = 1$. If $H$ is any subgroup of $G$ or if $G/N$ is any homomorphic image of $G$, then $U(H)$ and $U(G/N)$ also satisfy $w = 1$.

Proof. The result for $H$ is obvious since $U(H) \subseteq U(G)$. For $G/N$, let us suppose first that $N$ is a finite $p$-group. Then the kernel of the epimorphism $K[G] \to K[G/N]$ is a nilpotent ideal and therefore $U(G)$ maps onto $U(G/N)$. With this, it is clear that $U(G/N)$ satisfies $w = 1$. Next, let $N$ be a finite $p'$-group, and set $e = N/[N]$, where $N$ is the sum of the elements of $N$ in $K[G]$. Then $e$ is a central idempotent of $K[G]$ and $eK[G] \cong K[G/N]$. As a consequence, $U(G/N)$ is isomorphic to a subgroup of $U(G)$, and therefore $U(G/N)$ also satisfies $w = 1$.

Now suppose that $N$ is merely finite. Since $N'$ is a $p$-group, it follows that $N$ has a normal Sylow $p$-subgroup $P$. But then, $G/N = (G/P)/(N/P)$, so the result follows here by applying the preceding two special cases in turn. The general case is now a consequence of the fact that $G$ is locally finite. Indeed, if $\bar{u}_1, \ldots, \bar{u}_m$ are units of $K[G/N]$, then there exists a finite subgroup $L$ of $G$ such that these units and their inverses are in the image of $K[L]$. But, by the above, we know that $U(L)$ and $U(L/(N \cap L))$ both satisfy $w = 1$, so we conclude that $w(\bar{u}_1, \ldots, \bar{u}_m) = 1$, as required. □

It is convenient to record the following well known observation.

Lemma 2.2. Let $A$ be a normal abelian subgroup of $G$ and suppose that $G/A$ is cyclic of finite order $q$. If $G = \langle A, t \rangle$, then $G' = \langle A, t \rangle = \{ (a, t) \mid a \in A \}$ and $G' \cap \mathbb{C}_G(t)$ has period dividing $q$. 

Proof. Since $A$ is abelian, the map $a \mapsto (a, t) = a^{-1}a^t$ is easily seen to be an endomorphism of $A$ with image $(A, t)$ as given above. Note that $(A, t)$ is normalized by $A$ and by $t$, so $(A, t) \triangleleft G$. Now we can mod out by $(A, t)$ and, since $G/(A, t)$ is center-by-cyclic, it follows that this factor group is abelian and therefore that $(A, t) = G'$. Finally, if $b = (a, t) \in (A, t) \cap C_G(t)$, then
\[ b^q = b^{1+t+\cdots+t^{q-1}} = a^{(t-1)(1+t+\cdots+t^{q-1})} = a^{q-1} = 1 \]
since $t^q \in A$. \qed

We now come to the heart of the argument.

Lemma 2.3. Suppose that $G = \langle A, t \rangle$ where $A$ is a normal abelian subgroup and where $t$ has order $q$. If $U(G)$ satisfies a group identity, then $G'$ has finite period.

Proof. We proceed by induction on $q$, implicitly using Lemmas 2.1 and 2.2 throughout. Suppose first that $(t)$ has a proper subgroup $\langle s \rangle$. Then $H = \langle A, s \rangle$ is a subgroup of $G$ with the same structure and, by induction, $H' = \langle A, s \rangle$ has finite period. Of course, $B = \langle H', s \rangle$ also has finite period, and note that $B$ is normalized by both $A$ and $t$ since $(A, s) = H' \subseteq B$. Now consider $G = G/B = \langle A, \bar{t} \rangle$. Since $\bar{t}$ has smaller order than that of $t$, induction again implies that $G'$ has finite period. Thus $G'/(B \cap G')$ has finite period and the result clearly follows. In other words, it suffices to assume that $(t)$ has no proper subgroup and therefore that $q$ is a prime. Furthermore, we can assume that $t \notin A$ since otherwise $G = A$ is abelian and $G' = \langle 1 \rangle$. This implies that $G = A \rtimes \langle t \rangle$ is the semidirect product of $A$ by $\langle t \rangle$.

Suppose now that $G = A \rtimes \langle t \rangle$ and that $(t)$ has prime order $q$. Note that $t$ acts on $K[A]$ by conjugation and that this action determines a trace map from $K[A]$ to $K[A] \cap Z(K[G])$ given by
\[ tr(\sigma) = \sigma + \sigma^t + \cdots + \sigma^{t^{q-1}} \text{ for all } \sigma \in K[A]. \]

Here, of course, $Z(K[G])$ denotes the center of $K[G]$. Observe that $tr(\sigma)^p = tr(\sigma^p)$ and that, if $\zeta \in K[A] \cap Z(K[G])$, then $tr(\sigma \zeta) = tr(\sigma)\zeta$ and $tr(\zeta) = q\zeta$. Furthermore, if we set
\[ \tau = 1 + t + \cdots + t^{q-1} \in K[G], \]
then
\[ t^q = \tau, \quad t\sigma \tau = tr(\sigma)\tau \quad \text{and} \quad \tau^2 = q\tau. \]

By [GJV, Proposition 1] with $b = c$, it follows that if $\alpha, \beta \in K[G]$ with $\alpha^2 = 0 = \beta^2$, then $(\alpha\beta)^n = 0$ for some integer $n$ depending on the word $w$. We can, of course, assume that $n = p^k$ is a fixed power of $p$. There are now two cases to consider.

Case 1. $q \neq p$.

Proof. Let $a \in A$ and observe that $\alpha = \tau a^{-1}(1-t^{-1})$ has square 0 since $(1-t^{-1})\tau = 0$. Furthermore, $qa - tr(a)$ has trace 0, so it follows that $\beta = (qa - tr(a))\tau$ also has
square 0. Thus, by the above mentioned result of [GJV], we have \((\alpha \beta)^n = 0\) with \(n = p^k\). Now \(\text{tr}(a)\) is central and \((1 - t^{-1})\tau = 0\), so

\[
\alpha \beta = \tau a^{-1}(1 - t^{-1}) \cdot (qa - \text{tr}(a))\tau = \tau a^{-1}(1 - t^{-1})qa\tau
\]

\[
= q\tau(1 - a^{-1}a't^{-1})\tau = q\tau(1 - a^{-1}a't)\tau = q(q - \text{tr}(b))\tau
\]

where \(b = a^{-1}a' = (a, t)\). Thus, since \(q(q - \text{tr}(b))\) is central and \(\tau^2 = q\tau\), we have

\[
0 = (\alpha \beta)^{p^k} = q^{p^k} (q - \text{tr}(b))^{p^k} \tau^{p^k}
\]

\[
= q^{p^k} (q^{p^k} - \text{tr}(b^{p^k}))q^{p^k - 1} \tau = q(q - \text{tr}(b^{p^k}))\tau
\]

and hence \(q = \text{tr}(b^{p^k})\). Finally, note that the group element 1 occurs in the support of the left-hand side of the latter equation, so it must also occur in the right-hand expression. But all group elements in \(\text{tr}(b^{p^k})\) are conjugate to \(b^{p^k}\), and therefore we conclude that \(1 = b^{p^k} = (a, t)^{p^k}\), as required. \(\square\)

**Case 2.** \(q = p\).

**Proof.** Again let \(a \in A\) and note that both \(\tau\) and \(a^{-1}\tau a\) have square 0 since \(q = p\). Consider \(\alpha = a^{-1}\tau a \cdot \tau = a^{-1} \text{tr}(a)\tau\) and observe that

\[
\alpha^2 = a^{-1} \text{tr}(a)\tau \cdot a^{-1} \text{tr}(a)\tau = \text{tr}(a^{-1}) \text{tr}(a) \cdot a^{-1} \text{tr}(a)\tau = \text{tr}(a^{-1}) \text{tr}(a)\alpha.
\]

Thus, by induction, we have

\[
\alpha^i = [\text{tr}(a^{-1}) \text{tr}(a)]^{i-1} \alpha,
\]

and hence, if \(p^k\) is as above, then

\[
0 = \tau \alpha^{p^k} = [\text{tr}(a^{-1}) \text{tr}(a)]^{p^k - 1} \tau \cdot \alpha
\]

\[
= [\text{tr}(a^{-1}) \text{tr}(a)]^{p^k - 1} \tau \cdot a^{-1} \text{tr}(a)\tau = [\text{tr}(a^{-1}) \text{tr}(a)]^{p^k} \tau.
\]

Therefore we conclude that

\[
0 = [\text{tr}(a^{-1}) \text{tr}(a)]^{p^k} = \text{tr}(b^{-1}) \text{tr}(b)
\]

where \(b = a^{p^k}\).

Now observe that

\[
0 = \text{tr}(b^{-1}) \text{tr}(b) = (b^{-1} + b^{-t} + \cdots + b^{-t^{p-1}})(b + b^t + \cdots + b^{t^{p-1}})
\]

\[
= \sum_{i=0}^{p-1} \text{tr}(b^{-1}b^{t^i})
\]
and hence, since \( \text{tr}(b^{-1}b^t) = p1 = 0 \), we have

\[
0 = \sum_{i=1}^{p-1} \text{tr}(b^{-1}b^t) = \sum_{i \neq j} b^{-t_i}b^{t_j}.
\]

Note that the right-hand expression above is the sum of \( p(p - 1) \) formally different group elements. Thus, for this sum to be zero in \( K[G] \), these support elements must be equal in groups of size \( p \). In particular, they can take on at most \( p - 1 \) distinct values. But the conjugates of \( b^{-1}b^t \) which appear in \( \text{tr}(b^{-1}b^t) \) are either all equal or they take on \( p \) distinct values and, as we just observed, the latter situation cannot occur. Thus \( b^{-1}b^t \in (A, t) \cap C_G(t) \) and the preceding lemma yields

\[
1 = (b^{-1}b^t)^p = (a^{-p^k}a^{p^kt})^p = (a^{-1}a^t)^{p^{k+1}} = (a, t)^{p^{k+1}}.
\]

In other words, every element of \( G' = (A, t) \) has period dividing \( p^{k+1} \), and the lemma is proved. \( \square \)

The remainder of the proof is now routine and quite quick.

**Lemma 2.4.** \((i) \Rightarrow (ii)\).

**Proof.** Here we assume that \( U(G) \) satisfies a group identity, so [GSV, Theorem] and [P, Corollary 5.3.10] imply that \( G \) has a normal \( p \)-abelian subgroup \( A \) of finite index. Furthermore, by [GSV, Lemma 2.3], \( G' \) is a \( p \)-group. Thus the goal here is to show that \( G' \) has bounded period. Since \( A' \) is a finite normal \( p \)-subgroup of \( G \), it clearly suffices to consider \( G/A' \), or equivalently we can assume that \( A \) is abelian. Now let

\[
B = \langle L' \mid A \subseteq L \subseteq G \text{ and } L/A \text{ is cyclic} \rangle.
\]

Then, by Lemma 2.3, \( B \) is a subgroup of \( A \) generated by a finite number of groups each of finite period. Hence, since \( A \) is abelian, \( B \) also has finite period. Furthermore, \( B \triangleleft G \) and \( B \subseteq G' \). Finally, observe that \( A/B \) is a central subgroup of \( G/B \) of finite index, and therefore \( G/B \) has a finite commutator subgroup by [P, Lemma 4.1.4]. In other words, \( G'/B \) is finite, and this obviously implies that \( G' \) has finite period. \( \square \)

**§3. The Implication (ii) \Rightarrow (iii)**

The goal now is to show that the group theoretic conditions of part (ii) imply that \( U(G) \) satisfies a particular group identity, namely \( 1 = (x, y)^{p^k} = (x^{-1}y^{-1}xy)^{p^k} \) for some \( k \geq 0 \). If \( R \) is any \( K \)-algebra, let \( U(R) \) denote its group of units. For convenience, we first observe
Lemma 3.1. Let $R$ be a $K$-algebra and let $I$ be an ideal of $R$ which is nil of bounded degree $\leq p^k$. If $U(R/I)$ satisfies $(x, y)^{p^j} = 1$, then $U(R)$ satisfies $(x, y)^{p^j + k} = 1$.

Proof. The map $\bar{\cdot}: R \to R/I$ yields a group homomorphism $\bar{\cdot}: U(R) \to U(R/I)$ which is not necessarily onto. If $x, y \in U(R)$, then $\bar{x}, \bar{y} \in U(R/I)$, so $(\bar{x}, \bar{y})^{p^j} = 1$. Hence $(x, y)^{p^j} - 1 \in I$, so this element is nilpotent of degree $\leq p^k$, and consequently

$$(x, y)^{p^j + k} - 1 = [(x, y)^{p^j} - 1]^p = 0,$$

as required. □

Next, we need

Lemma 3.2. Let $A$ be a normal abelian subgroup of $G$ of finite index $n$ and let $I$ be a $G$-stable ideal of $K[A]$ which is nil of bounded degree $\leq p^k$. Then $I : K[G]$ is an ideal of $K[G]$ which is nil of bounded degree $\leq np^k$.

Proof. Let $g_1, g_2, \ldots, g_n$ be coset representatives for $A$ in $G$ and, for any $\alpha \in K[G]$, write $g_i \alpha = \sum_j \alpha_{i,j} g_j$ with $\alpha_{i,j} \in K[A]$. Then we know from [P, Lemma 5.1.10] that the map $\alpha \mapsto [\alpha_{i,j}]$ is an algebra embedding of $K[G]$ into the ring $M_n(K[A])$ of $n \times n$ matrices over the commutative algebra $K[A]$. Furthermore, if $\alpha \in I : K[G]$, then $g_i \alpha \in I : K[G]$, so each $\alpha_{i,j} \in I$. In other words, $I : K[G]$ embeds in $M_n(I)$. As a consequence, it suffices to show that $M_n(I)$ is nil of bounded degree $\leq np^k$. To this end, let $\tau \in M_n(I)$. Then $\tau$ satisfies its characteristic polynomial, so

$$\tau^n = \gamma_0 + \gamma_1 \tau + \cdots + \gamma_{n-1} \tau^{n-1}$$

for suitable scalars $\gamma_i \in I$. Thus, since all these elements commute and since $\gamma_i^{p^k} = 0$, we have

$$\tau^{np^k} = \gamma_0^{p^k} + \gamma_1^{p^k} \tau^{p^k} + \cdots + \gamma_{n-1}^{p^k} \tau^{np^k} = 0,$$

and the result follows. □

Finally, we can prove

Lemma 3.3. (ii) $\Rightarrow$ (iii).

Proof. By assumption, $G$ has a normal $p$-abelian subgroup $A$ of finite index and $G'$ is a $p$-group of bounded period $\leq p^j$. The goal is to show that $U(G)$ satisfies an identity of the form $(x, y)^{p^k} = 1$ for some $k \geq 0$. To start with, $A' \triangleleft G$ and we know that the kernel of the homomorphism $K[G] \to K[G/A']$ is nilpotent since $A'$ is a finite $p$-group. Thus, by Lemma 3.1, it suffices to consider $G/A'$, or equivalently we can assume that $A$ is abelian.
Next, if \( B = (A, G) \), then \( B \) is a normal subgroup of \( G \) contained in \( A \cap G' \). Thus \( B \) is a \( p \)-group of period \( \leq p^j \) and, since \( A \) is abelian, it is easy to see that the kernel \( I \) of the homomorphism \( K[A] \to K[A/B] \) is nil of bounded degree \( \leq p^j \). Indeed, if \( b_i \in B \) and if \( \alpha_i \in K[A] \), then

\[
\left[ \sum_i (1 - b_i)\alpha_i \right]^{p^j} = \sum_i (1 - b_i^{p^j})\alpha_i^{p^j} = 0
\]

since \( b_i^{p^j} = 1 \). Furthermore, \( I \) is a \( G \)-stable ideal of \( K[A] \) and therefore, by Lemma 3.2, \( I \cdot K[G] \) is a nil ideal of \( K[G] \) of bounded degree. In particular, since \( I \cdot K[G] \) is the kernel of the homomorphism \( K[G] \to K[G/B] \), it suffices to consider \( G/B \), or equivalently we can now assume that \( A \) is central.

Finally, it follows from [P, Lemma 4.1.4] that \( G' \) is finite and therefore that \( G \) is a \( p \)-abelian group. Again, this implies that the kernel of the map \( K[G] \to K[G/G'] \) is nilpotent so, by Lemma 3.1, we can now assume that \( G' = \{1\} \). But then \( K[G] \) is a commutative algebra, and consequently \( U(G) \) satisfies \( (x, y) = 1 \). With this, the lemma is proved.

Since the implication \((iii) \Rightarrow (i)\) is trivial, Lemmas 2.4 and 3.3 combine to yield Theorem 1.1.

REFERENCES


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