Answers to the Algebra Qualifying Exam
August 1999

1. (a) One can almost just say that since $H$ normalizes $G$ and $K$, it must normalize $C_G(K)$. More properly, let $g \in C_G(K)$, $k \in K$, $h \in H$. Then $gk = kg$, so conjugating by $h$ yields $g^hk^h = k^hg^h$. But $K \triangleleft H$, so $K^h = K$ and hence $k^h$ is a typical element of $K$. It follows that $g^h$ centralizes $K$, so $g^h \in C_G(K)$ and hence $C_G(K)^h \subseteq C_G(K)$. Thus $H$ normalizes $C_G(K)$.

(b) It follows from (a) that $\bar{G} = HC_G(K)$ is a group and that $C_G(K)\triangleleft \bar{G}$. Furthermore, since $H \triangleleft \bar{G}$ we have $H \triangleleft \bar{G}$. Note that $H \cap C_G(K) = \bar{C}_H(K) = \langle 1 \rangle$ by assumption. Thus $H$ and $C_G(K)$ are disjoint normal subgroups of $\bar{G}$, so these groups commute elementwise and hence $H$ centralizes $C_G(K)$.

2. (a) Since $Q$ is not prime there exist ideals $A', B'$ with $A' \not\subseteq Q$, $B' \not\subseteq Q$ and $A'B' \subseteq Q$. Let $A = A' + Q$ and $B = B' + Q$. Then $A$ and $B$ are ideals which properly contain $Q$ and $AB = (A' + Q)(B' + Q) \subseteq A'B' + Q = Q$.

(b) Suppose that $Q$ is not prime and by (a) choose ideals $A$ and $B$ which properly contain $Q$ and satisfy $AB \subseteq Q$. By the maximal nature of $Q$, there exist integers $m$ and $n$ with $A \supseteq I^m$ and $B \supseteq I^n$. Thus $Q \supseteq AB \supseteq I^{m+n}$, contradicting the given property of $Q$. Consequently, $Q$ must be prime.

(c) Let $S$ be the set of ideals of $R$ which do not contain any power of $I$. Since $I$ is not nilpotent, it follows that $0 \not\in S$ and hence $S$ is nonempty. Since $R$ satisfies the ascending chain condition on ideals, it satisfies the maximal condition. Thus $S$ contains a maximal member, say $Q$. By (b), this ideal $Q$ is prime and since $Q \in S$ it follows that $Q$ does not contain $I$.

3. (a) We know that $E = K[f(x)]$ is the splitting field over $K$ of a separable polynomial $f(x) \in K[x]$. Furthermore, $F = K[g(x)]$. Thus clearly $L = EF = K[f(x)g(x)]$ and note that every irreducible factor of $f(x)g(x) \in K[x]$ divides $f(x)$ or $g(x)$. It follows that $f(x)g(x)$ is also separable and, consequently, $L$ is Galois over $K$.

(b) Let $T = \text{Gal}(L/K)$. Since $E/K$ is Galois, it follows that every automorphism in $T$ must stabilize $E$. Thus, by restriction, we obtain a group homomorphism $\theta: T = \text{Gal}(L/K) \to \text{Gal}(E/K) = G$. Similarly, restriction yields a homomorphism $\phi: T = \text{Gal}(L/K) \to \text{Gal}(F/K) = H$. By combining these we obtain a map $(\theta \times \phi): T \to G \times H$ given by $(\theta \times \phi)(t) = (\theta(t), \phi(t)) \in G \times H$ for all $t$ in $T$. Now, if $t$ is in the kernel of this combined map, then $\theta(t) = 1$, so $t$ fixes $E$ elementwise. Similarly $t$ fixes $F$ elementwise. Thus $t$ fixes $EF = L$ elementwise and hence $t = 1$. Thus $\ker(\theta \times \phi) = \langle 1 \rangle$ so $\theta \times \phi$ is a one-to-one map and $T$ is isomorphic to its image, a subgroup of $G \times H$. 

4. (a) The characteristic polynomial of the matrix \( A(x) = \begin{bmatrix} 1 & -2 \\ 8 & x \end{bmatrix} \) in the variable \( \lambda \) is \( \phi(\lambda) = \det(\lambda I - A(x)) = \lambda^2 - (x + 1)\lambda + (x + 16) \). Since the complex numbers are algebraically closed this polynomial factors into linear factors and we know that \( A(x) \) will be diagonalizable if \( \phi \) has distinct roots. So we need only be concerned with the possibility that the two roots are equal. In this case, the discriminant is 0 so \( (x + 1)^2 = 4(x + 16) \). Hence \( 0 = x^2 - 2x - 63 = (x - 9)(x + 7) \) and \( x = 9 \) or \(-7\). In the latter two cases, \( A(x) \) has all eigenvalues equal. In particular, if it were diagonalizable, it would be similar to a scalar matrix and hence be scalar, a contradiction. Thus \( x = 9 \), \(-7\) are really exceptions.

(b) Notice that \( J^2 = nJ \), so \( J \) satisfies the polynomial \( \lambda^2 - n\lambda = \lambda(\lambda - n) \) which has the distinct roots 0 and \( n \). This implies that \( J \) is diagonalizable. Indeed, it will be similar to a diagonal matrix \( D \) having diagonal entries equal to 0 or \( n \) only, say \( a \) entries are 0 and \( b \) entries equal \( n \). Of course \( a + b = n \). Furthermore, since similar matrices have the same trace, we get \( n = \text{tr} J = \text{tr} D = a0 + bn \). Thus \( b = 1 \), \( a = n - 1 \) and \( J \) is similar to \( D = \text{diag}(n,0,0,\ldots,0) \).

An alternate argument is as follows. Let \( \theta: F[x,y] \to F \) be the evaluation map determined by \( x \mapsto \alpha \), \( y \mapsto \beta \). Then both \( f(x) \) and \( g(y) \) are in the kernel of \( \theta \), so \( I \subseteq \ker \theta \). But \( 1 \notin \ker \theta \), so \( 1 \notin I \) and \( I \neq F[x,y] \).

(b) Since \( x = f(x) + \alpha \) and \( y = g(y) + \beta \), it follows that any polynomial in \( x \) and \( y \) is a multiple of \( f(x) \) plus a multiple of \( g(y) \) plus an element of \( F \). Thus \( F[x,y] = I + F \) and by (a) we know that \( I \cap F = 0 \). In other words, \( F[x,y]/I \cong F \) is a field and hence \( I \) is a maximal ideal.

An alternate argument is as follows. Let \( \theta: F[x,y] \to F \) be the evaluation map determined by \( x \mapsto \alpha \), \( y \mapsto \beta \), and let \( J = \ker \theta \). Then \( F[x,y]/J \cong F \), a field, so \( J \) is a maximal ideal and clearly \( J \supseteq I \). On the other hand, note that \( x \equiv \alpha \mod I \) and \( y \equiv \beta \mod I \). Thus if \( h(x,y) \in F[x,y] \), then \( h(x,y) \equiv h(\alpha,\beta) \mod I \). In particular, if \( h(x,y) \in J \), then \( h(x,y) \equiv h(\alpha,\beta) = 0 \mod I \), so \( h(x,y) \in I \). Thus \( I \supseteq J \), and \( I = J \) is maximal.