1. (a) Let $H = \langle X, Y \rangle$ be the subgroup of $G$ generated by $X$ and $Y$. Certainly $X$ and $Y$ form a weird pair for $H$. We claim that $X$ and $Y$ are normal in $H$. If $y \in Y$, then $X^y$ is a subgroup of $H$ with $|X^y| = |X|$. Thus, by assumption, $X^y = X$ or $Y$. But $X^y = Y$ implies that $X = Y^{y^{-1}} = Y$, a contradiction. Thus $X^y = X$ and $y \in \mathbb{N}_H(X)$. It follows that $\mathbb{N}_H(X) \supseteq \langle X, Y \rangle = H$, so $X \triangleleft H$ and similarly $Y \triangleleft H$.

(b) Suppose, by way of contradiction, that $A \times 1$ and $1 \times B$ form a weird pair for $G = A \times B$. Then certainly $|A| = |A \times 1| = |1 \times B| = |B|$, and these orders are not equal to 1. Since $A$ is solvable, $A' \neq A$ and thus $A$ has a normal subgroup $N$ of index $p$ for some prime $p$. Then $p$ divides $|A| = |B|$, so $B$ has a subgroup $P$ of order $p$. The combined map $\theta: A \to A/N \cong P \to B$ is then a nontrivial homomorphism from $A$ to $B$. Let $C = \{(a, \theta(a)) | a \in A\} \subseteq A \times B = G$. Then it is easy to see that $C$ is a subgroup of $G$ different from $A \times 1$ and $1 \times B$. Furthermore, $C \cong A$ via the projection to the first coordinate. Thus $|C| = |A \times 1| = |1 \times B|$, contradicting the definition of weird pair.

(c) Let $G$ be solvable, by way of contradiction, that $X$ and $Y$ form a weird pair for $G$. By part (a), we can assume that $X$ and $Y$ are normal in $G$ and that $G = XY$. If $N = X \cap Y$, then $N \triangleleft G$ and it is easy to see that $A = X/N$ and $B = Y/N$ form a weird pair for $G/N$. Furthermore, $A, B \triangleleft G/N$, $A \cap B = 1$ and $G/N = AB$. Thus $G/N$ is the internal direct product of $A$ and $B$, and since $A$ is solvable, this contradicts the result of part (b).

2. (a) By assumption, $V$ has a composition series $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V$ of length $n$. Let $W_i = V_i/V_{i-1}$ for $i = 1, \ldots, n$, so that each $W_i$ is a simple $R$-module. Thus $P_i = \{r \in R | W_i r = 0\}$ is a primitive ideal of $R$ and we set $I = \bigcap_{i=1}^n P_i$. By definition of $J = \text{Jrad}(R)$, we know that $I \subseteq J$. Since $I \subseteq P_i$, we have $W_i I = 0$ and hence $V_i I \subseteq V_{i-1}$. It follows by induction that $V_i I^i \subseteq V_0 = 0$, so $V I^n = V_n I^n = 0$. Since $R$ acts faithfully on $V$, this yields $I^n = 0$. But $\text{Jrad}(R)$ contains all nilpotent ideals of $R$, so $J \supseteq I$ and we conclude that $J = I$. In particular, $J = I$ is an intersection of the $n$ primitive ideals $P_i$, and $J^n = I^n = 0$.

(b) Let $K$ be a field and let $R$ be the subring of the $2 \times 2$ matrix ring $M_2(K)$ with $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Then $R$ acts faithfully (on the right) on $V = (K, K) = K^2$, the $K$-vector space of $1 \times 2$ row vectors. If $V_1 = (0, K) \subseteq V$, then $V_1$ is an $R$-submodule of $V$ and $0 \subseteq V_1 \subseteq V$ is a composition series of length 2. This follows since $R \geq K$ implies that any $R$-module $W$ with $\dim_K W = 1$ must be irreducible. Finally, $\text{Jrad}(R) \neq 0$ since $I = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ is a nonzero nilpotent ideal of the ring $R$.

3. (a) Let the roots of $f(x)$ be $\alpha, 2\alpha, \beta_1, \ldots, \beta_k$. Since $f$ is monic, each of these elements is an algebraic integer, and their product is $\pm f(0)$, since $f(0)$ is the constant term of the polynomial. In particular, if $f(0) = 1$, then $\alpha \cdot 2\alpha \cdot \prod_{i=1}^k \beta_i = \pm 1$, so we have
1/2 = \pm \alpha^2 \prod_{i}^{k} \beta_i. Thus 1/2 is a noninteger rational number that is an algebraic integer. This is a contradiction since \( \mathbb{Z} \) is a UFD and hence integrally closed. The latter means that the only elements of the rationals \( \mathbb{Q} \) integral over \( \mathbb{Z} \) are the elements of \( \mathbb{Z} \) itself.

(b) Let \( K \) be the splitting field of \( f(x) \) over \( \mathbb{Q} \) and let \( G \) be the Galois group of \( K/\mathbb{Q} \). Since \( f \) is irreducible, we know that \( G \) is finite and that \( G \) permutes the roots of \( f \) transitively. In particular, there exists \( \sigma \in G \) with \( \sigma(\alpha) = 2\alpha \). Then, by induction, \( \sigma^n(\alpha) = 2^n\alpha \). But \( \sigma \) has finite order, say \( m \geq 1 \), so \( 2m\alpha = \sigma^m(\alpha) = \text{Id}(\alpha) = \alpha \). Thus \((2^m - 1)\alpha = 0 \) and since \( K \) has characteristic 0, we conclude that \( \alpha = 0 \).

4. (a) Since the complex numbers are algebraically closed, all eigenvalues of complex matrices are contained in the complex numbers. Now we know that \( \langle v, w \rangle = v^*w \) defines an Hermitian inner product on the space of complex \( n \times 1 \) column vectors. In particular, \( \langle v, v \rangle \) is always real and nonnegative. Now let \( \lambda \) be an eigenvalue of \( A^*A \) with corresponding eigenvector \( v \). Then \( A^*Av = \lambda v \), so \( v^*A^*Av = \lambda v^*v \). Note that \( v^*A^*Av = \langle Av, Av \rangle \geq 0 \) and that \( v^*v = \langle v, v \rangle > 0 \) since \( v \neq 0 \). Thus \( \lambda = \langle Av, Av \rangle / \langle v, v \rangle \) is real and nonnegative.

(b) If \( \lambda \) is an eigenvalue of \( I + A^*A \) with corresponding eigenvector \( v \), then \( v + A^*Av = (I + A^*A)v = \lambda v \), so \( A^*Av = (\lambda - 1)v \). Hence \( \lambda - 1 \) is an eigenvalue of \( A^*A \), so (a) implies that \( \lambda - 1 \geq 0 \). Thus \( \lambda \) is real and positive. Since \( \det(I + A^*A) \) is the product of the eigenvalues of \( I + A^*A \), we conclude that \( \det(I + A^*A) \) is real and positive.

5. (a) Since \( S \neq 0 \), we know that \( W = \{v \in V \mid vS = 0 \} \) is a subspace of \( V \) different from \( V \). Furthermore, \( S \in F[T] \) implies that \( S \) and \( T \) commute. Thus \( vS = 0 \) yields \( (vT)S = (vS)T = 0T = 0 \), so \( WT \subseteq W \) and the hypothesis implies that \( W = 0 \).

(b) \( F[T] \) is certainly a commutative ring acting faithfully on \( V \). If \( 0 \neq S \in F[T] \), then we know from part (a) that \( S \) is a nonsingular linear transformation and hence that \( S^{-1} \) exists. Since \( S \) satisfies its characteristic polynomial, we have \( S^r + a_1S^{r-1} + \cdots + a_rI = 0 \) where \( a_i \in F \) and \( r = \text{dim}_F V \). Furthermore, \( a_r \neq 0 \) since \( S \) is nonsingular. We can now multiply the polynomial equation by \( S^{-1} \) and \( a_r^{-1} \) to obtain

\[
S^{-1} = -(a_r^{-1}S^{r-1} + a_r^{-1}a_1S^{r-2} + \cdots + a_r^{-1}a_{r-1}I) \in F[T].
\]

Hence \( F[T] \) is a field.

(c) We know that \( F[T] \) acts faithfully on \( V \), so \( V \) is a vector space over this larger field. Since any subspace of \( V \) is \( T \)-stable, the hypothesis implies that \( V \) must be 1-dimensional over \( F[T] \). Thus if \( 0 \neq v \in V \), then \( V = vF[T] \cong F[T] \) as right \( F[T] \)-vector spaces and hence as right \( F \)-vector spaces. By definition of the degree of a field extension, we conclude that \( \text{dim}_F V = \text{dim}_F F[T] = |F[T] : F| \).