1. Since $A$ contains no nonidentity normal subgroup, it contains no nonidentity central element. In particular, if $1 \neq a \in A$, then $C_G(A) \neq G$. But $A$ is abelian, so $C_G(a) \supseteq A$. Hence, since $A$ is maximal, we have $C_G(a) = A$ for all $1 \neq a \in A$.

(i) Suppose $|A|$ is divisible by $p$ and let $x \in A$ have order $p$. Choose a Sylow $p$-subgroup $P$ of $G$ with $x \in P$. Then $A = C_G(x) \supseteq \mathbb{Z}(P)$. But $\mathbb{Z}(P) \neq 1$ since $P \neq 1$, so we can choose $1 \neq z \in \mathbb{Z}(P) \subseteq A$. Then $A = C_G(z) \supseteq P$ and consequently $|G : A|$ must be prime to $p$, a contradiction since $|G : A| = p^n \neq 1$.

(ii) Since $1 \neq A \subseteq N_G(A) \subseteq G$ and since $A$ is not normal in $G$, it follows from the maximality of $A$ that $A = N_G(A)$. Furthermore, if $A^x$ and $A^y$ are conjugates of $A$ having a nonidentity element $b$ in common, then $C_G(b) \supseteq \langle A^x, A^y \rangle$. But $A^x$ and $A^y$ are maximal and $b$ is not central, so $A^x = C_G(b) = A^y$. In other words, the $p^n = |G : A| = |G : N_G(A)|$ distinct conjugates of $A$ are disjoint. It follows that $|S| = |\bigcup_{x} A^x \setminus 1| = p^n(|A| - 1) = |G| - p^n$, so $|G \setminus S| = p^n$.

(iii) Let $P$ be a Sylow $p$-subgroup of $G$. Since $p$ does not divide $|A|$ and $p^n = |G : A|$, we see that $|P| = p^n$. Furthermore, $P$ is disjoint from the set $S$. Thus $P \subseteq G \setminus S$ and, since both of these sets have the same size, we conclude that $P = G \setminus S$. But $G \setminus S$ is clearly a normal subset of $G$, and hence $P$ is a normal subgroup of $G$.

2. (i) Note that the $R$-submodules of $R_R$ are precisely the right ideals of $R$. If $R$ is a division ring and if $I$ is a nonzero submodule of $R_R$, then $I$ is a right ideal containing a unit and hence $I = R$. Thus $R_R$ is simple. Conversely, suppose $R_R$ is simple and let $0 \neq x \in R$. Then $xR$ is a nonzero submodule of $R_R$, so $xR = R$ and there exists $y \in R$ with $xy = 1$. Similarly, since $y \neq 0$, there exists $z \in R$ with $yz = 1$. Thus $x = x(yz) = (xy)z = z$, so $yx = yz = 1$ and $y = x^{-1}$. Therefore $R$ is a division ring.

(ii) If $R$ is a division ring and if $V$ is a nonzero right $R$-module, choose $0 \neq v \in V$. Then the map $\theta: R \to V$ given by $\theta(r) = vr$ is an $R$-module homomorphism and its kernel is a proper right ideal of $R$. In particular, $ker \theta = 0$, so $\theta$ is one-to-one and $V \cong \theta(R) \cong R_R$. Conversely, suppose every nonzero $R$-module contains an isomorphic copy of $R_R$ and let $M$ be a maximal right ideal of $R$. Then $V = R/M$ is an irreducible $R$-module which, by assumption, contains $R_R$. Thus $V = R_R$, so $R_R$ is irreducible and consequently $R$ is a division ring by part (i).

3. We use the fact that if $E \subseteq \mathbb{C}$ is a finite Galois extension of $\mathbb{Q}$ and if $\theta: E \to \mathbb{C}$ is a field homomorphism, then $\theta(E) = E$ and $\theta$ restricts to a field automorphism of $E$.

(i) Let $F \subseteq \mathbb{C}$ be a finite Galois extension of $\mathbb{Q}$ with $\alpha, \beta \in F$ and $E \subseteq F$. Say $F$ is the splitting field of the polynomial $g(x) \in \mathbb{Q}[x]$. Since $\alpha$ and $\beta$ are roots of the same irreducible polynomial $f(x) \in \mathbb{Q}[x]$, there exists a field isomorphism $\sigma: \mathbb{Q}[\alpha] \to \mathbb{Q}[\beta]$ with $\sigma(\alpha) = \beta$. Note that $F$ is the splitting field of $g(x)$ over $\mathbb{Q}[\alpha]$ and it is the splitting field of $g(x)$ over $\mathbb{Q}[\beta]$. Thus, since $\sigma$ fixes $g(x) \in \mathbb{Q}[x]$, we see that $\sigma$ extends to a field isomorphism $\tau: F \to F$. Indeed, since $E \subseteq F$ and $E$ is Galois over $\mathbb{Q}$, we see that $\tau$ restricts to an automorphism of $E$. In particular, $\tau: \mathbb{Q}[\alpha] \cap E \to \mathbb{Q}[\beta] \cap E$. Since $\tau^{-1}$ maps $E$ to $E$.
and \(\mathbb{Q}[\beta]\) to \(\mathbb{Q}[\alpha]\), we have \(\tau^{-1}: \mathbb{Q}[\beta] \cap E \rightarrow \mathbb{Q}[\alpha] \cap E\). Thus \(\tau: \mathbb{Q}[\alpha] \cap E \rightarrow \mathbb{Q}[\beta] \cap E\) is the required field isomorphism.

(ii) If \(E = \mathbb{Q}[\varepsilon]\), where \(\varepsilon\) is a root of unity, then we know that \(G = \text{Gal}(E/\mathbb{Q})\) is abelian. Thus every subgroup of \(G\) is normal and hence every intermediate field is Galois over \(\mathbb{Q}\). In particular, \(\mathbb{Q}[\alpha] \cap E\) is Galois over \(\mathbb{Q}\) and since \(\tau: \mathbb{Q}[\alpha] \cap E \rightarrow \mathbb{Q}[\beta] \cap E\) is an isomorphism, we must have \(\tau(\mathbb{Q}[\alpha] \cap E) = \mathbb{Q}[\alpha] \cap E\) and hence \(\mathbb{Q}[\alpha] \cap E = \mathbb{Q}[\beta] \cap E\).

4. (i) Say \(\dim_F V = n\) and let \(\alpha_1, \alpha_2, \ldots, \alpha_n\) be the \(n\) distinct roots of the characteristic polynomial of \(S\). Then these are the \(n\) eigenvalues of \(S\) with corresponding eigenvectors \(v_1, v_2, \ldots, v_n\). We know that \(v_1, v_2, \ldots, v_n\) form a basis for \(V\) and that the only eigenvectors for \(S\) are contained in the \(n\) lines \(Fv_1, Fv_2, \ldots, Fv_n\). Since \(S\) and \(T\) commute, we have \((v_i T) S = (v_i S) T = (\alpha_i v_i) T = \alpha_i (v_i T)\). Thus \(v_i T\) is either 0 or an \(\alpha_i\)-eigenvector for \(S\). But \(\alpha_i\) occurs with multiplicity 1 as a root of the characteristic polynomial for \(S\), so the \(\alpha_i\)-eigenspace has dimension 1 and hence \(v_i T \in Fv_i\). In other words, \(v_i T = \beta_i v_i\) for some \(\beta_i \in F\) and hence each \(v_i\) is an eigenvector for \(T\).

(ii) If \(T\) is nilpotent, then each \(\beta_i\) must be 0, so \(v_i T = 0\). But \(V\) is spanned by \(v_1, v_2, \ldots, v_n\), so \(VT = 0\) and hence \(T = 0\).

5. (i) Let \(\theta_i: V \rightarrow V/M_i\) be the natural epimorphism with \(\ker \theta_i = M_i\) and let \(\theta: V \rightarrow W\) be given by \(\theta(v) = (\theta_1(v), \theta_2(v), \ldots, \theta_n(v))\). Then \(\theta(v) = 0\) if and only if \(\theta_i(v) = 0\) for all \(i\). Thus \(\ker \theta = \bigcap_i \ker \theta_i = \bigcap_i M_i = 0\), by assumption. In other words, \(\theta\) is one-to-one and hence \(V \cong \theta(V) \subseteq W\).

(ii) Write \(W_i = V/M_i\) so that \(W = W_1 \oplus W_2 \oplus \cdots \oplus W_n\), and note that

\[(*) \quad 0 \subseteq W_1 \subseteq W_1 \oplus W_2 \subseteq \cdots \subseteq W_1 \oplus W_2 \oplus \cdots \oplus W_n = W\]

is an ascending series of submodules of \(W\). Since \((W_1 \oplus \cdots \oplus W_i)/(W_1 \oplus \cdots \oplus W_{i-1}) \cong W_i\) is a simple \(R\)-module, we see that \((*)\) is a composition series for \(W\) of length \(n\). Furthermore, \(V\) is isomorphic to a submodule of \(W\), so \(V\) and all its submodules have composition series of length \(\leq n\). We consider the composition factors for \(V\). Since the series \(0 \subseteq M_i \subseteq V\) can be refined to a composition series, we see that \(V\) has a composition factor isomorphic to \(V/M_i\) for each \(i\). But these irreducible modules \(V/M_i\) are given to be all nonisomorphic. Hence \(V\) must have at least \(n\) distinct composition factors and the composition length of \(V\) is \(\geq n\). As we saw, \(W\) has composition length precisely \(n\) and \(W\) contains an isomorphic copy \(\theta(V)\) of \(V\). In particular, \(W/\theta(V)\) must have composition length 0, so \(W = \theta(V) \cong V\), as required.