1. Let $G$ be a group of order $2^4 \cdot 3^3 \cdot 11$ and let $H$ be a group of order $5^3 \cdot 11$.
   a. Show that $H$ has a normal Sylow 11-subgroup. (2 points)
   b. If the number of Sylow 5-subgroups of $G$ is (strictly) less than 16, prove that $G$ has a proper normal subgroup of order divisible by 5. (4 points)
   c. If $G$ has exactly sixteen Sylow 5-subgroups, show that $G$ has a normal Sylow 11-subgroup. (4 points)

2. Let $R$ be a (not necessarily commutative) ring with 1 and suppose that $R$ can be written as the sum $R = \sum_{i=1}^{m} I_i$, where the $I_i$ are finitely many (two-sided) ideals of $R$ satisfying $I_i \cap I_j = 0$ whenever $i \neq j$.
   a. Prove that, for every simple right $R$-module $M$, there exists a unique subscript $k$ such that $MI_k \neq 0$. (5 points)
   b. Show that if $i \neq j$, then every right $R$-module homomorphism $\theta: I_i \to I_j$ is the zero map. (5 points)

3. Let $L/K$ be a finite degree Galois extension of fields with Galois group given by $\text{Gal}(L/K) = G$, and let $E$ be an intermediate field. Then $E$ is said to be a 2-tower over $K$ if there exists a chain of fields $K = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ such that $|E_i : E_{i-1}| = 2$ for all $i = 1, 2, \ldots, n$.
   a. If $G$ is abelian, prove that $E$ is a 2-tower over $K$ if and only if the degree $|E : K|$ is a power of 2. (7 points)
   b. Show by example that the characterization of 2-towers given in part (a) is false if $G$ is allowed to be a nonabelian group. (3 points)

4. Let $A$ be an $n \times n$ matrix over the complex numbers and assume that the rank of $A$ is equal to 1.
   a. What are the possible Jordan canonical forms for $A$? Justify your answer. (5 points)
   b. For each of the forms obtained in part (a), compute the characteristic polynomial of $A$ and the minimal polynomial of $A$. (5 points)

5. Let $R = F[x, y]$ be the polynomial ring over the field $F$ in the two indeterminates $x$ and $y$, and let $I = xR$ be the principal ideal of $R$ generated by $x$. Define $S = F + I$, so that $S$ is a subring of $R$, and observe that $I$ is an ideal of $S$.
   a. Show that $I$ is not finitely generated as an ideal of $S$. (5 points)
   b. Prove that there are infinitely many ideals of $S$ that are not ideals of $R$. (5 points)