1. Let $N$ be a normal subgroup of the finite group $G$ and suppose that $G/N$ is a $p$-group for some prime $p$.
   a. If $N \subseteq Z(G)$, the center of $G$, show that the commutator subgroup $G'$ of $G$ is a $p$-group. (5 points)
   b. Now assume that $N$ is cyclic (but not necessarily central in $G$). Prove that $N \cap G' \subseteq Z(G')$ and deduce that $G''$ is a $p$-group. (5 points)

2. Let $R$ be a commutative integral domain with 1. A nonzero, nonunit element $s \in R$ is said to be “special” if, for every element $a \in R$, there exist $q, r \in R$ with $a = qs + r$ and such that $r$ is either 0 or a unit of $R$.
   a. If $s \in R$ is special, prove that the principal ideal $(s)$ generated by $s$ is maximal in $R$. (3 points)
   b. Show that every polynomial in $\mathbb{Q}[X]$ of degree 1 is special in $\mathbb{Q}[X]$. (2 points)
   c. Prove that there are no special elements in the polynomial ring $\mathbb{Z}[X]$. (Hint. Apply the definition of special with $a = 2$ and with $a = X$.) (5 points)

3. Let $F$ be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, where $F/\mathbb{Q}$ is a finite Galois extension. Let $\alpha \in F$ and let $f(X) \in \mathbb{Q}[X]$ be its minimal monic polynomial. Assume that $1 = |\alpha|$, the absolute value of $\alpha$, and that $\text{Gal}(F/\mathbb{Q})$ is abelian.
   a. Show that $F$ is closed under complex conjugation. (2 points)
   b. Prove that $|\beta| = 1$ for every complex root $\beta$ of $f(X)$. (3 points)
   c. Writing $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$, show that $|a_i| \leq 2^n$ for all $i$ with $0 \leq i < n$. (2 points)
   d. Prove that $F$ contains only finitely many algebraic integers having absolute value 1 and deduce that each of these is a root of unity. (3 points)
4. Let $V$ be vector space over the field $K$ and let $(\ , \): V \times V \to K$ be a bilinear form on $V$.
   a. If $V$ is finite dimensional and if $W$ is a proper subspace of $V$, show that there exists a nonzero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$. (5 points)
   b. Now let $V$ have an infinite basis $B$ and let $(\ , \)$ be the unique bilinear form such that, for all $a, b \in B$, we have $(a, b) = 0$ if $a \neq b$ and $(a, b) = 1$ if $a = b$. If $W$ is the subspace of $V$ spanned by all vectors of the form $a - b$ with $a, b \in B$, show that $W$ is a proper subspace of $V$ and that there is no nonzero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$. (5 points)

5. Let $R$ be a ring with 1. We say that a right $R$-module $W$ is “infinitely generated” if it is not finitely generated as an $R$-module.
   a. Let $V$ be a right $R$-module and let $W$ be a submodule of $V$. If $W$ is infinitely generated, prove that there exists a submodule $M$ with $W \subseteq M \subseteq V$ such that $M$ is infinitely generated, but such that all submodules of $V$ properly containing $M$ are finitely generated. (5 points)
   b. If $R$ is right Noetherian, show that $M = V$ in the above situation. (2 points)
   c. If $R$ is not right Noetherian, show that it is possible to choose $V$ and $W$ as in part (a) so that $M \neq V$. (3 points)