1. A finite group is said to be \( \textit{perfect} \) if it has no nontrivial abelian homomorphic image. 
   i. Show that a perfect group has no nontrivial solvable homomorphic image. (3 points)
   
2. Let \( R \) be a ring and let \( V \) be a right \( R \)-module. Assume that every simple submodule of \( V \) is a direct summand of \( V \).
   i. If \( W \) is any submodule of \( V \), show that any simple submodule of \( W \) is a direct summand of \( W \). (5 points)
   
3. Let \( \alpha \) be the real positive 16th root of 3 and consider the field \( F = \mathbb{Q}[\alpha] \) generated by \( \alpha \) over the rationals \( \mathbb{Q} \). Notice that we have the chain of intermediate fields

\[
\mathbb{Q} \subseteq \mathbb{Q}[\alpha^8] \subseteq \mathbb{Q}[\alpha^4] \subseteq \mathbb{Q}[\alpha^2] \subseteq \mathbb{Q}[\alpha] = F.
\]

   i. Compute the degrees of these five intermediate fields over \( \mathbb{Q} \) and conclude that these fields are all distinct. (4 points)
   
4. Let \( X \) be a subspace of \( M_n(\mathbb{C}) \), the \( \mathbb{C} \)-vector space of all \( n \times n \) complex matrices. Assume that every nonzero matrix in \( X \) is invertible. Prove that \( \dim_{\mathbb{C}} X \leq 1 \).

5. Let \( E \) be an algebraic extension of the rational numbers \( \mathbb{Q} \) and let \( \alpha \in E \).
   i. Prove that there exists a nonzero integer \( n \in \mathbb{Z} \) such that \( n\alpha \) is an algebraic integer. (4 points)
   
   ii. Show that \( \mathbb{Z}[\alpha] \) does not contain \( \mathbb{Q} \) and hence conclude that \( \mathbb{Z}[\alpha] \) is not a field. (6 points)