1. Fix a prime \( p \) and let \( G \) be a finite group with the property that every nonidentity \( p \)-subgroup of \( G \) is contained in a unique Sylow \( p \)-subgroup of \( G \). Suppose \( N \trianglelefteq G \) and \( |N| \) is divisible by \( p \).
   i. If \( P \) and \( Q \) are Sylow \( p \)-subgroups of \( G \), show that \( Q = P^n \) for some element \( n \in N \). (6 points)
   ii. Prove that \( G/N \) has a unique Sylow \( p \)-subgroup. (4 points)

2. Let \( R \) be a commutative domain and write \((a)\) for the principal ideal generated by \( a \in R \). Recall that an element of \( R \) is said to be irreducible if it is nonzero, not a unit, and has no proper factorization.
   i. Show that \((a) \subseteq (b)\) if and only if \( b \mid a \), and that \((a) = (b)\) if and only if \( b = au \) for some unit \( u \in R \). (2 points)
   ii. If \( R \) is a UFD (unique factorization domain), prove that the set of principal ideals of \( R \) satisfies the maximal condition. (4 points)
   iii. If the set of principal ideals of \( R \) satisfies the maximal condition, show that every nonzero, nonunit element of \( R \) can be written as a finite product of irreducible elements. (4 points)

3. Let \( p \) be a prime, let \( F \subseteq K \) be fields of characteristic 0, and assume that \( F \) contains a primitive \( p \)-th root of unity. Fix \( a \in K \).
   i. Prove that there exists a field \( E \supseteq K \) such that \( E \) contains a \( p \)-th root of \( a \) and \( |E : K| = 1 \) or \( p \). (4 points)
   ii. Now assume that \( K \) is a finite degree Galois extension of \( F \). Show that there exists a field \( E \supseteq K \) such that \( E \) contains a \( p \)-th root of \( a \), \( E \) is Galois over \( F \), and \( |E : K| \) is a power of \( p \). (6 points)

4. Let \( V \) be a finite dimensional vector space over a field of characteristic 0. Suppose \( T: V \to V \) is a linear operator such that the trace \( \text{tr} T^k = 0 \) for all integers \( k \geq 1 \).
   i. Show that the constant term of the characteristic polynomial of \( T \) is zero, and deduce that \( T(V) \neq V \). (5 points)
   ii. Let \( S \) denote the restriction of \( T \) to the subspace \( T(V) \), so that \( S \) is a linear operator on \( T(V) \). Prove that \( \text{tr} S^k = 0 \) for all integers \( k \geq 1 \). (4 points)
   iii. Show that \( T \) is nilpotent. (1 point)

5. Let \( G \) be a (not necessarily finite) group and let \( \theta: G \to G \) be a homomorphism such that \( \theta^n(G) = \{1\} \) for some integer \( n \geq 1 \).
   i. If the kernel of \( \theta \) is finite, prove that the kernel of \( \theta^2 \) is finite, and deduce that \( G \) is finite. (5 points)
   ii. If \( \theta(G) \) has finite index in \( G \), prove that \( \theta^2(G) \) has finite index in \( G \), and deduce that \( G \) is finite. (5 points)