Let $\triangle ABC$ be a triangle and $H_A$, $H_B$, $H_C$ be the feet of the altitudes from $A$, $B$, $C$ respectively. The triangle $\triangle H_AH_BH_C$ is called the orthic triangle (some authors call it the pedal triangle) of $\triangle ABC$. We denote the orthocenter by $H$; it is the point of concurrence of the three altitudes. The incenter of a triangle is the center of its inscribed triangle. It is equidistant from the three sides and is the point of concurrence of the angle bisectors.

**Theorem.** The orthocenter $H$ of $\triangle ABC$ is the incenter of the orthic triangle $\triangle H_AH_BH_C$.

**Proof.** Because $\angle AH_AC = 90^\circ$, $\angle CAH = \angle CAH_A$, $\angle ACB = \angle ACH_A$, we have that $\angle CAH = 90^\circ - \angle ACB$. Because the quadrilateral $H_BAH_CH$ has right angles at $H_A$ and $H_B$ it is cyclic, in fact $H_B$ and $H_C$ lie on the circle with diameter $AH$. Hence $\angle H_BAH = \angle H_BHC$ as they subtend the same arc on this circle. But $\angle H_BAH = \angle CAH$ so $\angle H_BHC = 90^\circ - \angle ACB$. The same argument (reading $A$ for $B$) shows that also $\angle H_AHC = 90^\circ - \angle ACB$. Hence $\angle H_BHC = \angle H_AHC$, i.e. the line $HH_C$ bisects $\angle H_BHC$. By the same reasoning $HH_A$ bisects $\angle H_BHA$ and $HH_B$ bisects $\angle H_CB$.\[\Box\]
Theorem. Let the altitudes \( AH, BH, CH \) meet the circumcircle of triangle \( \triangle ABC \) respectively in \( X, Y, Z \). Then \( H \) is the incenter of \( \triangle XYZ \).

Proof. As in the previous proof \( \angle CAX \) (i.e. \( \angle CAH_A \)) is the complement of \( \angle ACB \). But \( \angle CAX = \angle CZX \) as they subtend the same arc on the circumcircle. Hence \( \angle CZX \) is the complement of \( \angle ACB \). The same argument (reversing the roles of \( A \) and \( B \)) shows that \( \angle CZY \) is the complement of \( \angle ACB \). Hence \( \angle CZX = \angle CZY \), i.e. \( CZ \) bisects \( \angle XZY \). The same argument applies at the other vertices \( X \) and \( Y \). By hypothesis the lines \( AX, BY, CZ \) are the altitudes of \( \triangle ABC \) and we have just shown that they are the angle bisectors of \( \triangle XYZ \). Hence their common point \( H \) is both the orthocenter of \( \triangle ABC \) and the incenter of \( \triangle XYZ \).

Corollary. \( H_CH = H_CZ, H_BH = H_BY, H_AH = H_AX \).

Proof. By what we have already proved and the principle that equal angles subtend equal arcs we have that \( \angle XAB = \angle XYB = \angle ZYB = \angle ZAB \) so the right triangles \( \triangle AH_CH \) and \( \triangle AH_CZ \) are congruent. Hence \( H_CH = H_CZ \). The other equalities follow in the same way.

Remark. In other words \( X, Y, Z \) are the reflections of \( H \) in the sides \( BC, CA, AB \) respectively. Note also that the triangle \( \triangle XYZ \) is similar to the orthic triangle \( \triangle H_AH_BH_C \) as corresponding sides make equal angles with the altitudes (e.g. \( \angle HH_CH_B = 90^\circ - \angle ACB = \angle HZY \)), so they are parallel.