These notes summarize the key points in the first chapter of *Differential Geometry of Curves and Surfaces* by Manfredo P. do Carmo. I wrote them to assure that the terminology and notation in my lecture agrees with that text. *All page references in these notes are to the Do Carmo text.*

1. **Definition.** A *parameterized smooth curve* is a map \( \alpha : I \rightarrow \mathbb{R}^n \) where \( I \subseteq \mathbb{R} \) is an interval. The set theoretic image

\[
C = \alpha(I) := \{ \alpha(t) : t \in I \}
\]

is called the *trace* of \( \alpha \) and \( \alpha \) is called a *parameterization* of \( C \). See do Carmo page 2.

2. **Remark.** For do Carmo the words *differentiable* and *smooth* are synonymous. I prefer the word *smooth*. The adjective *differentiable* is often omitted by do Carmo.

3. **Remark.** On page 2 do Carmo says that the interval \( I \) should be open but on page 30 he extends the notion of smoothness to closed intervals. A function defined on a closed interval \([a, b]\) is said to be *smooth* iff it extends to an open interval containing \([a, b]\). This means that the derivatives of the function are defined at the end points \( a \) and \( b \).

4. **Definition.** A *reparameterization* of \( \alpha : I \rightarrow \mathbb{R}^n \) is a smooth map \( \beta : J \rightarrow \mathbb{R}^n \) of form \( \beta = \alpha \circ \sigma \) where \( \sigma : J \rightarrow I \) is a diffeomorphism. That \( \sigma \) is a diffeomorphism means \( \sigma \) is one-to-one and onto (so there is an inverse map \( \sigma^{-1} : J \rightarrow I \) and that \( \sigma'(t) \neq 0 \) for \( t \in I \) (so that the map \( \sigma^{-1} \) is also smooth).

5. **Remark.** If \( \beta \) is a reparameterization of \( \alpha \), then the maps \( \alpha \) and \( \beta \) have the same trace \( C \). The idea of the definition is that we should think of \( \alpha \) and \( \beta \) as different ways of describing the same curve \( C \). However do Carmo avoids giving a precise definition of an (unparameterized) curve.

6. **Example.** The circle \( C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \) is the trace of the parameterized curve \( \alpha : \mathbb{R} \rightarrow \mathbb{R}^2 \) defined by

\[
\alpha(\theta) = (\cos \theta, \sin \theta) = (\cos(\theta + 2\pi), \sin(\theta + 2\pi)).
\]
Define a map $\beta : \mathbb{R} \to \mathbb{R}^2$ by
$$\beta(t) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right).$$

This map is a reparameterization of the restriction of $\alpha$ to the open interval $(-\pi, \pi)$ as follows:
$$\left(\cos(2\varphi), \sin(2\varphi)\right) = \left(\frac{\cos^2 \varphi - \sin^2 \varphi}{\cos^2 \varphi + \sin^2 \varphi}, \frac{2\sin \varphi \cos \varphi}{\cos^2 \varphi + \sin^2 \varphi}\right) = \left(\frac{1 - \tan^2 \varphi}{1 + \tan^2 \varphi}, \frac{2 \tan \varphi}{1 + \tan^2 \varphi}\right).$$

Take $2\varphi = \theta$, $t = \tan \varphi = \tan(\theta/2)$, and we get $\alpha = \beta \circ \sigma$ where $\sigma(\theta) = \tan(\theta/2)$. The common trace of (the restriction of) $\alpha$ and the map $\beta$ is the punctured circle $C \setminus (-1, 0)$. (This particular reparameterization is called the Weierstrass substitution or half angle substitution. It is one of the main techniques used to evaluate integrals in calculus.)

7. **Definition.** Let $\alpha : I \to \mathbb{R}^n$ be a smooth parameterized curve. The derivative $\alpha'(t)$ is called the *velocity vector* at $t$. The map $\alpha$ is called *regular* iff its velocity vector never vanishes. The map $\alpha$ is said to be *parameterized by arc length* iff its tangent vector always has length one.

8. **Theorem.** A smooth regular parameterized curve $\alpha$ has a reparameterization by arc length, i.e., there is a reparameterization $\beta : J \to \mathbb{R}^n$ of $\alpha$ such that $|\beta'(s)| = 1$ for $s \in J$.

**Proof:** This is the content of Remark 2 in do Carmo page 21. The reparameterization is defined by $\beta = \alpha \circ \sigma$ where $\sigma$ is a solution of the differential equation
$$\sigma'(s) = \frac{1}{|\alpha'(\sigma(s))|}.$$  
By the chain rule $\beta'(s) = \alpha'(\sigma(s))\sigma'(s)$ so $|\beta'(s)| = 1$.

9. **Remark.** The arc length
$$\ell(C) = \int_a^b |\alpha'(t)| \, dt$$
of the trace $C$ of a regular parameterized curve $\alpha : [a, b] \to \mathbb{R}^n$ is independent of the parameterization $\alpha$ used to define it. This is an easy consequence of the formula for changing variables in a definite integral: if $\sigma : [a, b] \to [c, d]$ is a diffeomorphism, then
$$\int_a^b |\alpha'(t)| \, dt = \int_c^d |\alpha'(\sigma(r))| |\sigma'(r)| \, dr.$$  
(The change of variables is $t = \sigma(r)$ so $dt = \sigma'(r) \, dr$.) When $\alpha$ is parameterized by arc length, $\ell(C) = |b - a|$.

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10. The reparameterization in Theorem 8 is unique in the following sense: If
\( \beta_1 : J_1 \to \mathbb{R}^n \) and \( \beta_2 : J_2 \to \mathbb{R}^n \) are two reparameterizations of the same map \( \alpha \) then \( \beta_2 = \beta_1 \circ \sigma \) where \( \sigma : J_2 \to J_1 \) has one of the two forms \( \sigma(s) = s + c \) or \( \sigma(s) = -s + c \). (This is because \( |\sigma'(s)| = 1 \).) On page 6 do Carmo says that when \( \sigma(s) = -s + c \) the two curves \( \beta_1 \) and \( \beta_2 \) are said to **differ by a change of orientation**.

This use of the word *orientation* can be viewed as a special case of the definition of *orientation of a vector space* that do Carmo gives on pages 11 and 12. For a regular curve \( \alpha \) the one dimensional vector space \( \mathbb{R} \alpha'(t) \subseteq \mathbb{R}^n \) is called the **tangent space** to the curve at the point \( \alpha(t) \). The velocity vector \( \alpha'(t) \) is a basis for this space. Changing the orientation of the curve changes the sign of the velocity vector \( \alpha'(t) \) and thus reverses the orientation of the tangent space.

11. **Remark.** Note the distinction between the tangent space and the tangent line. The tangent line is the line containing the points \( \alpha(t) \) and \( \alpha(t) + \alpha'(t) \). (See do Carmo page 5.) This line need not pass through the origin of \( \mathbb{R}^n \) and thus is not a vector subspace of the vector space \( \mathbb{R}^n \). This illustrates the difference between points and vectors.

12. **Remark.** The two orientations of \( \mathbb{R}^3 \) correspond to the thumb, forefinger, and middle finger of the right and left hands. (Recall the right hand rule from calculus.) The two orientations of \( \mathbb{R}^2 \) correspond to **clockwise** and **counter clockwise**. The two orientations of \( \mathbb{R} = \mathbb{R}^1 \) correspond to the two directions **increasing** and **decreasing**.

13. **Definition.** A map \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) is called an *isometry* iff it preserves distance i.e. iff it satisfies
\[
|\Phi(p) - \Phi(q)| = |p - q|
\]
for \( p, q \in \mathbb{R}^n \). A map \( \mathbb{R}^n \to \mathbb{R}^n \) is called a **translation** iff there is a vector \( c \in \mathbb{R}^n \) such that the map sends the point \( p \in \mathbb{R}^n \) to the point \( p + c \). A linear transformation \( \rho : \mathbb{R}^n \to \mathbb{R}^n \) is called **orthogonal** iff it satisfies \( \langle \rho u, \rho v \rangle = \langle u, v \rangle \) for all vectors \( u, v \in \mathbb{R}^n \). A **rigid motion** of \( \mathbb{R}^n \) is an isometry \( \Phi \) such that the corresponding orthogonal linear transformation \( \rho \) preserves orientation, i.e. has positive determinant.

14. **Theorem.** A map \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) is an isometry if and only if it is the composition of a translation and an orthogonal linear transformation.

**Proof:** For if see do Carmo page 23 Exercise 6 and do Carmo page 228 Exercise 7. The converse is not very difficult but is not needed in the rest of these notes so the proof is omitted.

15. **Theorem.** Let \( \alpha : [a, b] \to \mathbb{R}^n \) be smooth, \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) be an isometry, and \( \beta = \Phi \circ \alpha \). Then the curves \( \alpha \) and \( \beta \) have the same arc length. If \( \alpha \) is parameterized by arc length so is \( \beta \).

**Proof:** This is because \( \Phi \) preserves the length of vectors. The first part also follows from Exercise 8 on page 10 of do Carmo.
16. Definition. Let \( \alpha : I \to \mathbb{R}^n \) be parameterized by arc length. Then the **unit tangent vector** is the vector valued function \( t : I \to \mathbb{R}^n \) defined by

\[
t(s) = \alpha'(s) = \frac{d}{ds} \alpha(s),
\]
the **curvature vector** is the vector valued function \( I \to \mathbb{R}^n \)

\[
\alpha''(s) = \frac{d}{ds} t(s) = \frac{d^2}{ds^2} \alpha(s),
\]
and the **curvature** is the length \( \kappa \) of the curvature vector, i.e.

\[
\kappa(s) = |t'(s)| = |\alpha''(s)|.
\]

The **unit normal vector** is the normalized curvature vector

\[
n = \frac{t'}{|t'|}.
\]
(The vector \( n \) is defined only where the curvature \( \kappa \) is not zero.) The **binormal vector** is the vector product

\[
b = t \wedge n
\]
of the unit tangent vector \( t \) and the unit normal vector \( n \). (The binormal vector is defined only when \( n = 3 \).)

17. Theorem. Let \( \alpha : I \to \mathbb{R}^n \) be parametrized by arc length, \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) be an isometry, and \( \beta = \Phi \circ \alpha : I \to \mathbb{R}^n \). Then \( \beta \) is also parametrized by arc length and \( \alpha \) and \( \beta \) have the same curvature. If \( n = 3 \) and \( \Phi \) is a rigid motion they have the same torsion.

**Proof:** Exercise 6 page 23 of do Carmo.

18. Standing Assumption. Henceforth we assume that \( \alpha : I \to \mathbb{R}^3 \) is a regular curve parameterized by arc length.

19. Theorem. Then the vectors \( t, n, b \) are orthonormal, i.e.

\[
|t| = |n| = |b| = 1, \quad t \cdot n = t \cdot b = n \cdot b = 0.
\]

The ordered orthonormal basis \( t, n, b \) is called the **Frenet trihedron**.

**Proof:** (See do Carmo pages 18-19.) The equations \( |t| = |n| = 1 \) hold by definition. Since \( |t|^2 = t \cdot t \) is constant we get

\[
0 = \frac{d}{ds} |t|^2 = \frac{d}{ds} t \cdot t = 2 t \cdot t' = 2 \kappa t \cdot n
\]
so \( t \cdot n = 0 \). Now \( b \) is the vector product of two orthogonal unit vectors \( t \) and \( n \) so it is itself a unit vector and is orthogonal to both \( t \) and \( n \).
20. **Corollary.** The derivative $b'$ of the binormal vector $b$ is parallel to the unit normal vector $n$, i.e. there is a real valued function $\tau$ such that

$$b' = \tau n, \quad \tau = b' \cdot n.$$

The function $\tau$ is called the **torsion**.

**Proof:** Since $t' \wedge n = \kappa n \wedge n = 0$ we have

$$b' = (t \wedge n)' = t' \wedge n + t \wedge n' = t \wedge n'$$

so $b' \cdot t = b' \cdot n' = 0$. □

21. **Frenet Formulas.** The Frenet trihedron satisfies the differential equations

$$
\begin{align*}
    t' &= \kappa n, \\
    n' &= -\kappa t - \tau b, \\
    b' &= \tau n.
\end{align*}
$$

**Proof:** The first and last formulas hold by definition. For the middle formula differentiate the identities $n \cdot t = n \cdot b = 0$ and $n \cdot n = 1$ to get

$$
\begin{align*}
    0 &= n' \cdot t + n \cdot t' = n' \cdot t + \kappa n \cdot n = n' \cdot t + \kappa \\
    0 &= n' \cdot b + n \cdot b' = n' \cdot t + \tau n \cdot n = n' \cdot b + \tau \\
    0 &= 2n' \cdot n
\end{align*}
$$

Since the Frenet trihedron is orthonormal

$$n' = (n' \cdot t)t + (n' \cdot n)n + (n' \cdot b)b.$$

This proves the middle Frenet formula. □

22. **Remark.** The Frenet formulas may be written in matrix form as

$$
\begin{pmatrix}
    t' \\
    n' \\
    b'
\end{pmatrix} =
\begin{pmatrix}
    0 & \kappa & 0 \\
    -\kappa & 0 & -\tau \\
    0 & \tau & 0
\end{pmatrix}
\begin{pmatrix}
    t \\
    n \\
    b
\end{pmatrix}
$$

The coefficient matrix is skew symmetric. This is no coincidence. The two triples

$$t(s), n(s), b(s), \quad t(s_0), n(s_0), b(s_0)$$

are both bases for the vector space $\mathbb{R}^3$ so there is a unique change of basis matrix $U(s)$ satisfying

$$
\begin{pmatrix}
    t(s) \\
    n(s) \\
    b(s)
\end{pmatrix} = U(s)
\begin{pmatrix}
    t(s_0) \\
    n(s_0) \\
    b(s_0)
\end{pmatrix}.
$$

Since the two bases are both orthonormal, the matrix $U(s)$ is orthogonal. Differentiating with respect to $s$ and evaluating at $s = s_0$ gives the Frenet formula (in matrix form) evaluated at $s = s_0$. But $U(s_0)$ is the identity matrix and $U(s)^* = U(s)^{-1}$ so $U^*(s)U(s)$ is the identity matrix so differentiating at $s$ and evaluating at $s_0$ gives

$$U'(s_0) + U'(s_0) = 0.$$
23. **Theorem.** Reversing the orientation of \( \alpha \) leaves the curvature \( \kappa \) and the torsion \( \tau \) unchanged, i.e. if \( \beta(s) = \alpha(-s) \) the curves \( \alpha \) and \( \beta \) have the same curvature and torsion at \( s = 0 \).

**Proof:** By definition the curvature \( \kappa \) is nonnegative, the normal vector is only defined at points where the curvature \( \kappa \) is not zero, reversing the orientation of \( \alpha \) reverses the sign of the unit tangent vector \( t \) and leaves the sign of the curvature vector unchanged. Reversing the orientation of \( \alpha \) reverses the sign of \( t \), preserves the sign of \( n \), and therefore reverses the sign of \( b = t \wedge n \). But reversing the orientation of \( b \) reverses the sign of \( b' \) so reversing the orientation of \( \alpha \) preserves the sign of \( b' \) and hence (by the Frenet formula \( b' = \tau n \)) preserves the sign of \( \tau \).

24. **Fundamental Theorem.** Let \( \kappa, \tau : I \to \mathbb{R} \) be smooth functions defined on an interval \( I \). Assume \( \kappa > 0 \). Then

(Existence.) There is a curve \( \alpha : I \to \mathbb{R}^3 \) parameterized by arc length with curvature \( \kappa \) and torsion \( \tau \).

(Uniqueness.) If \( \alpha, \beta : I \to \mathbb{R}^3 \) are two curves parameterized by arc length both having curvature \( \kappa \) and torsion \( \tau \), then there is a rigid motion \( \Phi : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( \beta = \Phi \circ \alpha \).

**Proof:** See do Carmo page 309.

25. **Corollary.** The curvature and torsion of the helix \( \alpha(\theta) = (a \cos \theta, a \sin \theta, b \theta) \) are both constant so for any two points \( p \) and \( q \) on the helix there is a rigid motion carrying \( p \) to \( q \) and mapping the helix to itself.

26. **Gauss curvature.** In the case of a plane curve \( (n = 2) \) it is possible to choose a normal vector even when the curvature is zero. In this case since \( t \) and \( n \) are orthogonal unit vectors we can define \( n \) by rotating \( t \) clockwise through 90 degrees:

\[
\begin{align*}
t &= (\xi, \eta), & n &= (\eta, -\xi).
\end{align*}
\]

With this definition both \( t \) and \( n \) change sign when the orientation is reversed so to maintain the equation \( t' = \kappa n \) it is necessary to allow \( \kappa \) to be negative.

For a plane curve \( \alpha : I \to \mathbb{R}^2 \) parameterized by arc length we can view the unit normal vector as a map to the unit circle and define an angle \( \theta = \theta(s) \) by the formula

\[
n(s) = (\cos \theta(s), \sin \theta(s)).
\]

We then define the **signed curvature** by the formula

\[
\kappa = \frac{d\theta}{ds}.
\]

The signed curvature \( \kappa \) for a plane curve \( C \subseteq \mathbb{R}^2 \) is analogous to the Gauss curvature \( K \) of a surface \( S \subseteq \mathbb{R}^3 \). (See do Carmo pages 146, 155, 167.) Note
that when \( \alpha(s) = (\cos s, \sin s) \) is the counter clockwise parameterization of the unit circle in \( \mathbb{R}^2 \), the vector \( \mathbf{n} \) defined by rotation of \( \mathbf{t} \) as above is the outward normal (=radius vector) to the circle and the curvature \( \kappa \) is identically one. Thus the curvature compares the curve \( \alpha \) to the unit circle.

**27. Setup for local canonical form.** Assume that \( \alpha : I \to \mathbb{R}^3 \) has positive curvature and \( s_0 \in I \). The Taylor expansion

\[
\alpha(s) = \alpha(s_0) + (s - s_0)\alpha'(s_0) + \frac{(s - s_0)^2}{2}\alpha''(s_0) + \frac{(s - s_0)^3}{6}\alpha'''(s_0) + \cdots
\]

tells us what the trace \( C \) of \( \alpha \) looks like near the point \( \alpha(s_0) \in \mathbb{C} \). Because any reparameterization of \( C \) has the same trace we assume that \( \alpha \) is parameterized by arc length. Because the reparameterization defined by \( \sigma(s) = s - s_0 \) is also a parameterization by arc length, we assume that \( s_0 = 0 \). Because the arc length, curvature, and torsion are invariant under rigid motions, we assume that

\[
\alpha(0) = (0, 0, 0), \quad \mathbf{t}(0) = (1, 0, 0), \quad \mathbf{n}(0) = (0, 1, 0), \quad \mathbf{b}(0) = (0, 0, 1).
\]

**28. Local Canonical Form.** In the notation of Setup 27 above, the Taylor expansion of \( \alpha(s) = (x(s), y(s), z(s)) \) is

\[
x(s) = s - \frac{\kappa(0)s^3}{6} + R_x
\]

\[
y(s) = \frac{\kappa(0)s^2}{2} - \frac{\kappa'(0)s^3}{6} + R_y
\]

\[
z(s) = -\frac{\kappa(0)\tau(0)s^3}{6} + R_z
\]

where \( R_x, R_y, R_z = o(s^3) \).

**Proof:** There is no constant term in these formulas because \( \alpha(0) = 0 \). By definition \( \alpha' = \mathbf{t}, \quad \alpha'' = \kappa \mathbf{n} \).

Differentiating once more gives

\[
\alpha''' = \kappa' \mathbf{n} + \kappa \mathbf{n}' = \kappa' \mathbf{n} + \kappa(-\kappa \mathbf{t} - \tau \mathbf{b})
\]

by the second Frenet formula. Now evaluate at \( s = s_0 = 0 \).

**29. Application.** Recall (Remark 11 above and do Carmo page 5) that the **tangent line** to the trace \( C \) of a regular curve \( \alpha \) at a point \( p_0 = \alpha(s_0) \in C \) is the line containing the two points \( p_0 \) and \( p_0 + \mathbf{t}_0 \) where \( \mathbf{t}_0 = \mathbf{t}(s_0) \). The **osculating plane** to \( C \) at \( p_0 \) is the plane containing the three points \( p_0, p_0 + \mathbf{t}_0, p_0 + \mathbf{n}_0 \) where \( \mathbf{n}_0 = \mathbf{n}(s_0) \). (See do Carmo pages 17, 29, 30. The definition assumes that the curvature \( \kappa(s_0) \) at \( p_0 \) is positive.) Let \( p_1 = \alpha(s_1) \) and \( p_2 = \alpha(s_2) \) be two other points on \( C \) distinct from \( p_0 \) and each other. Then as \( s_1 \to s_0 \) the limit of the line through \( p_0 \) and \( p_1 \) is the tangent line at \( p_0 \) and the limit as \( s_1, s_2 \to s_0 \) of the plane through \( p_0, p_1, \) and \( p_2 \) is the osculating plane.