Complexity of binary trees of uncountable height

Reese Johnston, University of Wisconsin-Madison

http://www.math.wisc.edu/~rwjohnston

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Definitions

- Given a language $\mathcal{L}$ containing the symbol $\in$, an $\mathcal{L}$-formula $\varphi$ is $\Delta^0_0$ if all quantifiers appearing in $\varphi$ are bounded (i.e., of the form $\forall x \in y$ or $\exists x \in y$).
- $\varphi$ is $\Sigma^0_1$ if it is of the form $\exists x \psi$, where $\psi$ is $\Delta^0_0$.
- We operate within the universe $L_{\omega_1}$. A set $X \subseteq L_{\omega_1}$ is c.e. if $X$ is definable by a $\Sigma^0_1(L_{\omega_1})$ formula (a $\Sigma^0_1$ formula with parameters in $L_{\omega_1}$). $X$ is computable if both $X$ and $\overline{X}$ are c.e.
For the most part, we will rely on an uncountable version of the Church-Turing Thesis; we think of this as running a program that is allowed to manipulate countably infinite objects and run for any countable number of stages.
Facts

- There is a computable bijection between $\omega_1$ and the universe; this induces a computable well-order on the universe. Denote this $\prec_{\omega_1}$.
- For any degree $d \geq 0'$, there is a degree $a$ such that $a' = d$.
- For any degree $d \geq 0^{(n)}$, there is a degree $a$ such that $a^{(n)} = d$.

These, and many more, can be proven by the direct analogue of the proofs in $\omega$.
Trees

A binary tree is a subset of $2^{<\omega_1}$ (in the $\omega$-setting, $2^{<\omega}$) that is closed downward. A path is a string of length $\omega_1$ (in the $\omega$-setting, $\omega$) every initial segment of which is on the tree.

In $\omega$

In the $\omega$ setting, we have the Low Basis Theorem, which states that every computable tree has a low path, among numerous other results about the behaviour of $\Pi^0_1$ classes.

In $\omega_1$

Very little is known about $\Pi^0_1$ classes. Does the Low Basis Theorem still hold?
No. It is an easy observation that binary trees of uncountable height and $\omega_1$-branching trees are interchangeable, and by a result of Fokina, Friedman, Knight and Miller the problem of detecting paths through $\omega_1$-branching trees is $\Sigma^1_1$-complete.

The problem seems to be that in a binary tree of height $\omega_1$, Weak König’s Lemma fails; because of the opportunity for uncountable width, a node can have extensions at every height but not extend to a path.

Does the Low Basis Theorem hold for trees of countable width?
Theorem (J.)
There is a computable tree $T$ of countable width which has a unique path, and such that this path is Turing equivalent to $\emptyset'$.

Lemma
There is a computable Aronszajn tree; i.e., a tree of countable width and uncountable height, having no path.
Proof of Thm

We construct the tree, $T$, in stages, as $T = \bigcup_s T_s$, such that $T_s \setminus T_t \subseteq 2^s \setminus 2^t$. At each stage, we will have $P_s$, our “intended path”; the (usually false) assumption is that the path $P$ through the tree will begin with $P_s$. We will also have a collection $Q_s \subset T_s$; these will be identified as “roots of Aronszajn trees”, and will always have successor length. Fix a computable Aronszajn tree $A$. We have a sequence of strategies $R_\alpha$; the goal of $R_\alpha$ will be to “code” the $\alpha$ bit of $\emptyset'$ into $P$ in some recoverable sense.

At any stage, some of the $R_\alpha$ have been initialized and some have not. An uninitialized requirement cannot require attention. An initialized requirement has associated with it an ordinal $\beta_\alpha$ and a member $\sigma_\alpha$ of $Q_s$. We call $\beta_\alpha$ the branchpoint of $R_\alpha$. If $R_\alpha$ has been initialized and $\emptyset'_s(\alpha) \neq P_s(\beta_\alpha)$, then $R_\alpha$ requires attention.

At a successor stage $s + 1$, there are two cases.
Proof, continued

Case One: Some $R_\alpha$ requires attention. Let $\tau \in T_s \cap 2^s <_L$-least such that $\tau \ni \sigma_\alpha$; let $P_{s+1} = \tau$, and let $Q_{s+1} = (Q_s \setminus \{\sigma_\alpha\}) \cup \{P_s\}$. Deinitialize $R_\beta$ for all $\beta > \alpha$. Let $T_s^* = T_s$.

Case Two: No $R_\alpha$ requires attention. In this case, fix $\alpha$ least such that $R_\alpha$ is uninitialized, and initialize it. Set $\beta_\alpha = |P_s| + 1$, $\sigma_\alpha = P_s1$. Let $P_{s+1} = P_s0$, and let $Q_{s+1} = Q_s \cup \{P_s1\}$. Let $T_s^* = T_s \cup \{P_s0, P_s1\}$.

Regardless of which case occurs, we then proceed to the expansion phase.

Expansion Phase: For each $\sigma \in Q_s$, add to $T_s^*$ a collection of extensions of $\sigma$ so that the collection of extensions of $\sigma$ is naturally isomorphic to an initial segment of $A$. 
Proof, continued

Finally, we consider the limit stages: At a limit stage, we ignore all requirements, and instead simply expand. Let $P_s = \lim_{t \to s} P_t$ (by the claim below, this limit exists). Let $Q_s = \limsup_{t<s} Q_t$. For each $\sigma \in Q_s$, again extend to maintain the isomorphism with $A$.

This completes the construction.
Claim

Let $s$ a limit, and let $\alpha < s$. There exists a stage $t < s$ such that for all $u, v > t$, $P_u(\alpha) = P_v(\alpha)$. As a consequence (taking $s = \omega_1$) $P = \lim P_s$ is a $\Delta^0_2$ path through $T$.

The only way for $P(\alpha)$ to fail to stabilize is if, by stage $s$, there are infinitely many stages $u$ at which some strategy $R_\gamma$ with $\beta_\gamma < \alpha$ receives attention. In particular, there is a sequence $\{\gamma_n\}_{n \in \omega}$ such that $\beta_{\gamma_n} < \alpha$ and $R_{\gamma_{n+1}}$ receives attention at some stage after $R_{\gamma_n}$ does and before stage $s$, for all $n$. Note that these must receive attention after stage $\alpha$, otherwise $P(\alpha)$ would not yet have been defined. However, when $R_\gamma$ receives attention, it forces all lower-priority strategies to reinitialize and select a new branchpoint larger than the current stage - so it must be that $\{\gamma_n\}_{n \in \omega}$ is a descending sequence. Since this is a sequence of ordinals, it must stabilize. But no strategy can receive attention more than once; this is a contradiction, so no such sequence exists.
Claim

$P$ is the unique path in $T$.

Let $\sigma \in T$ not on $P$. Note that a string is extended at stage $s$ only if it is in one of the Aronszajn trees at stage $s$ or else is on $P_s$; so $\sigma$ extends to a path in $T$ only if it uncountably often meets one of these conditions. If $\sigma \preceq P_s$ uncountably often, then by the fact that $P$ stabilizes $\sigma \preceq P$, contradicting our hypothesis. But then $\sigma$ is in an Aronszajn tree uncountably often. But by construction, it is always the same Aronszajn tree (if the tree around $\sigma$ ceases to be Aronszajn, no prefix of $\sigma$ will be chosen to be a root of a new Aronszajn tree). So clearly there is no path through $\sigma$. 
Claim

$P$ computes $\emptyset'$.  

The bits of $\emptyset'$ are encoded in the branchpoints of the strategies, so it suffices to recover these branchpoints. By induction, $P$ can do so.
Thanks!