cosh and sinh

The hyperbolic functions cosh and sinh are defined by

\[ (1) \quad \cosh x = \frac{e^x + e^{-x}}{2} \]

\[ (2) \quad \sinh x = \frac{e^x - e^{-x}}{2} \]

We compute that the derivative of \( \frac{e^x + e^{-x}}{2} \) is \( \frac{e^x - e^{-x}}{2} \) and the derivative of \( \frac{e^x - e^{-x}}{2} \) is \( \frac{e^x + e^{-x}}{2} \), i.e.

\[ (3) \quad \frac{d}{dx} \cosh x = \sinh x \]

\[ (4) \quad \frac{d}{dx} \sinh x = \cosh x \]

Note that \( \sinh x > 0 \) for \( x > 0 \), and \( \sinh x < 0 \) for \( x < 0 \). However \( \cosh x \geq 0 \) for all \( x \) (strictly positive away from 0). \( \sinh x \) is increasing for all \( x \). \( \cosh x \) is increasing for all \( x > 0 \) (and decreasing for \( x < 0 \)). Note that \( \cosh(x) = \cosh(-x) \) and \( \sinh(-x) = -\sinh(x) \). The minimum of \( \cosh x \) is attained at \( x = 0 \) where \( \cosh(0) = 1 \), thus \( \cosh(x) \geq 1 \) for all \( x \).

Draw pictures:
The inverse of \( \sinh \)

\( \sinh x \) is (strictly) increasing and \( \lim_{x \to \infty} \sinh(x) = \infty \) and \( \lim_{x \to -\infty} \sinh(x) = -\infty \). We see that the range of \( \sinh \) is \( (-\infty, \infty) \) and \( \sinh \) is invertible. Then

\[
\sinh : (-\infty, \infty) \to (-\infty, \infty) \\
\sinh^{-1} : (-\infty, \infty) \to (-\infty, \infty)
\]

Let us compute the inverse. That is, we consider the equation \( \sinh(x) = y \), and express \( x \) in terms of \( y \). This means we need to solve for \( x \) in \( e^x - e^{-x} = 2y \). To do this we first set \( w = e^x \), determine \( w \) and then take the natural logarithm of \( w \). The equation for \( w \) becomes \( w - w^{-1} = 2y \) or \( w^2 - 2yw - 1 = 0 \). By the quadratic formula there are two possibilities for \( w \), namely \( w = y + \sqrt{y^2 + 1} \) and \( w = y - \sqrt{y^2 + 1} \). The first solution for \( w \) is positive, the second is negative, and since \( w = e^x \) has to be positive we can discard the negative solution for \( w \). Hence \( e^x = w = y + \sqrt{y^2 + 1} \) and after taking the logarithm we see that \( x = \ln(y + \sqrt{y^2 + 1}) \). We have thus computed the inverse function for \( \sinh \) and it is given by

\[
(5) \quad \sinh^{-1}(y) = \ln(y + \sqrt{y^2 + 1}).
\]

In some of the European literature this inverse function is denoted by \( \text{Arsinh} \), hence \( \text{Arsinh}(y) = \ln(y + \sqrt{y^2 + 1}) \). \(^1\)

The derivative of \( \sinh^{-1} \)

We could use the general formula for the derivative of inverse functions, or just the above formula for \( \sinh^{-1} \), let’s do the latter.

I am now writing \( x \) for the independent variable. Using the chain rule we get

\[
\frac{d}{dx}(\ln(x + \sqrt{x^2 + 1})) = \frac{1}{x + \sqrt{x^2 + 1}} \frac{d}{dx}(x + \sqrt{x^2 + 1})
\]

We note that \( \frac{d}{dx}(x + (x^2 + 1)^{1/2}) = 1 + \frac{1}{2}(x^2 + 1)^{-1/2}2x \) and get that the last displayed expression is equal to

\[
\frac{1}{x + \sqrt{x^2 + 1}} \left( 1 + \frac{x}{\sqrt{x^2 + 1}} \right) = \frac{1}{x + \sqrt{x^2 + 1}} \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}
\]

Therefore we get the rule

\[
(6) \quad \text{Arsinh}'(x) \equiv \frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{x^2 + 1}}.
\]

For the integral this is

\[
\int \frac{1}{\sqrt{x^2 + 1}} \, dx = \ln(x + \sqrt{x^2 + 1}) + C.
\]

\(^1\)\text{Ar sinh stands for Area sinus hyperbolicus. I have not seen this in any American textbook. Use sinh}^{-1} \text{ instead but then make sure that you do not confuse it with the reciprocal of sinh} y.
The inverse of $\cosh$

As a function on the real line $\cosh$ does not have an inverse (note that $\cosh(x) = \cosh(-x)$ so that two different points in $x$ correspond to the same value of $\cosh$). However, if we restrict the domain to $[0, \infty)$ then $\cosh$ is strictly increasing and invertible. The range of $\cosh$ is $[1, \infty)$ so that we have

$$\cosh : [0, \infty) \to [1, \infty)$$
$$\cosh^{-1} : [1, \infty) \to [0, \infty)$$

We compute $\cosh^{-1}(y)$ for $y \geq 1$. Thus, for each $y \geq 1$ we wish to determine an $x \geq 0$ so that $\cosh x = y$, or equivalently $(e^x + e^{-x})/2 = y$. To determine $x$ we again first determine $w = e^x$ from the equation $w + w^{-1} = 2y$, or equivalently, $w^2 - 2yw + 1 = 0$. This quadratic equation has two solutions, namely $w$ could be $y \pm \sqrt{y^2 - 1}$. The possibility of the minus sign can be discarded since a calculation\footnote{Indeed $y - \sqrt{y^2 - 1} = \frac{y^2 - (\sqrt{y^2 - 1})^2}{y + \sqrt{y^2 - 1}} = \frac{1}{y + \sqrt{y^2 - 1}} < 1$ for $y \geq 1$.} shows that $y - \sqrt{y^2 - 1}$ is $< 1$ for all $y \geq 1$, and therefore it can not be an $e^x$ for some $x > 0$. Thus $w = y + \sqrt{y^2 - 1}$ and $e^x = w$, so $\cosh^{-1} y = x = \ln w$. We get the formula

$$\cosh^{-1}(y) = \ln(y + \sqrt{y^2 - 1}). \tag{7}$$

Again this inverse function is occasionally denoted by $\text{Arcosh}$, so we may write sometimes $\text{Arcosh}(y) = \ln(y + \sqrt{y^2 + 1})$.

The derivative of $\cosh^{-1}$

Again I am now writing $x$ for the independent variable (it is now restricted to $x > 1$). The calculation is completely analogous to the calculation for the derivative of $\sinh^{-1}$. By the chain rule we get

$$\frac{d}{dx} \left( \ln(x + \sqrt{x^2 - 1}) \right) = \frac{1}{x + \sqrt{x^2 - 1}} \frac{d}{dx} (x + \sqrt{x^2 - 1})$$

and we now calculate that this yields $\frac{1}{\sqrt{x^2 - 1}}$. Hence we get the rule

$$\text{Arcosh}'(x) \equiv \frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2 - 1}}. \tag{8}$$

This also yields (for $x > 1$)

$$\int \frac{1}{\sqrt{x^2 - 1}} \, dx = \ln(x + \sqrt{x^2 - 1}) + C.$$
Problems

1. Prove the following identities.
   (i) \((\cosh x)^2 - (\sinh x)^2 = 1\).
   (ii) \(\cosh(2x) = (\cosh x)^2 + (\sinh x)^2\).
   (iii) \(\sinh(2x) = 2 \sinh x \cosh x\).

2. Compute the integrals
   (i) \(\int_0^x (a^2 - t^2)^{-1/2} \, dt\) for \(|x| < a\).
   (ii) \(\int_0^x (a^2 + t^2)^{-1/2} \, dt\) for all \(x\).
   (iii) \(\int_0^x (a^2 - t^2)^{1/2} \, dt\) for \(|x| < a\).
   (iv) \(\int_0^x (a^2 + t^2)^{1/2} \, dt\) for all \(x\).

3. The function \(\tanh\) is defined by
   \[
   \tanh x = \frac{\sinh x}{\cosh x}
   \]
   (i) Show that \(\tanh\) is defined and differentiable for all \(x\) and show that its derivative is given by
   \[
   \tanh'(x) = \frac{1}{\cosh^2 x}.
   \]
   (ii) Show that the range of \(\tanh\) is the interval \((-1, 1)\) and that
   \[
   \tanh : (-\infty, \infty) \to (-1, 1)
   \]
   is invertible.
   (iii) Prove that the inverse function
   \[
   \tanh^{-1} : (-1, 1) \to (-\infty, \infty)
   \]
   is given by \(\tanh^{-1}(y) = \frac{1}{2}(\ln(1 + y) - \ln(1 - y))\).
   (iv) Sketch the graph of \(\tanh\) and the graph of \(\tanh^{-1}\).