Taylor’s theorem

**Theorem 1.** Let \( f \) be a function having \( n+1 \) continuous derivatives on an interval \( I \). Let \( a \in I \), \( x \in I \). Then

\[
(*) \quad f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x, a)
\]

where

\[
(**) \quad R_n(x, a) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) \, dt.
\]

**Proof.** For \( n = 0 \) this just says that

\[
f(x) = f(a) + \int_a^x f'(t) \, dt
\]

which is the fundamental theorem of calculus.

For \( n = 1 \) we use the formula \((*)_0\) and integrate by parts. That is we apply the formula

\[
\int_a^x u(t)v'(t) \, dt = u(x)v(x) - u(a)v(a) - \int_a^x u'(t)v(t) \, dt
\]

with \( u(t) = f'(t) \), \( v(t) = t - x \).

We then get

\[
\int_a^x f'(t) \, dt = \left[ (t-x)f'(t) \right]_a^x - \int_a^x (t-x)f''(t) \, dt
\]

\[
= (x-a)f'(a) + \int_a^x (x-t)f''(t) \, dt.
\]

Therefore by \((*)_0, (**)_0\)

\[
f(x) = f(a) + \int_a^x f'(t) \, dt
\]

\[
= f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t) \, dt.
\]

We prove the general case using induction. We show that the formula \((*)_n\) implies the formula \((*)_n+1\). Suppose we have already proved the formula for a certain number \( n \geq 0 \). Then we integrate by parts in the remainder term \( R_n(x, a) \) (cf. the above formula with \( u(t) = f^{(n+1)}(t) \), \( v(t) = (x-t)^{n+1}/(n+1) \)). We obtain

\[
\int_a^x (x-t)^n f^{(n+1)}(t) \, dt = \left[ \frac{-(x-t)^{n+1}}{n+1} f^{(n+2)}(t) \right]_a^x - \int_a^x \frac{-(x-t)^{n+1}}{n+1} f^{(n+2)}(t) \, dt
\]

\[
= \frac{(x-a)^{n+1}}{n+1} f^{(n+1)}(a) + \int_a^x \frac{(x-t)^{n+1}}{n+1} f^{(n+2)}(t) \, dt.
\]
Deviding by \(n!\) yields
\[
R_n(x, a) = \frac{(x - a)^{n+1}}{(n+1)!} + R_{n+1}(x, a).
\]

Assuming the correctness of \((*)_n\), \((**)_n\) we may deduce \((*)_n+1\), \((**)_n+1\):
\[
f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \cdots + \frac{f^n(a)}{n!}(x - a)^n + R_n(x, a)
= f(a) + \frac{f'(a)}{1!}(x - a) + \cdots + \frac{f^n(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(x - a)^{n+1} + R_{n+1}(x, a). \tag*{□}
\]

with \(R_{n+1}(x, a) = \int_a^x \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)dt\).

We now want to estimate the remainder term \(R_n\).

**Theorem 2.** Let \(f\) be as in Theorem 1 and \(R_n\) as in \((**)_n\). Let
\[
M = \max\{ |f^{(n+1)}(t)| : t \text{ between } a \text{ and } x \}.
\]
Then
\[
|R_n(x, a)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}.
\]

**Proof.** Let \(I(a, x)\) be the interval with endpoints \(a\) and \(x\)
\[
|R_n(x, a)| \leq \int_{I(a, x)} \left| \frac{(x-t)^n}{n!} f^{(n+1)}(t) \right| dt
\leq \int_a^x \frac{(x-t)^n}{n!} M dt = \frac{M}{(n+1)!} |x - a|^{n+1}.
\]

The last theorem can be strengthened as follows.

**Theorem 3.** Let \(f\) be as in Theorem 1. There is a number \(\gamma\) between \(a\) and \(x\) such that
\[
R_n(x, a) = \frac{f^{(n+1)}(\gamma)}{(n+1)!} (x - a)^{n+1}
\]

**Proof.** Suppose first \(a < x\).

Let \(k\) be the minimum of \(f^{(n+1)}(t)\) in the interval \([a, x]\) (as above) and let \(K\) be the maximum of \(f^{(n+1)}\) in this interval. Then
\[
k \int_a^x \frac{(x-t)^n}{n!} dt \leq R_n(x, a) \leq K \int_a^x \frac{(x-t)^n}{n!} dt.
\]

Evaluating the integral (as above) and deviding by the integral yields
\[
\frac{k}{(n+1)!} \leq \frac{R_n(x, a)}{(x - a)^{n+1}} \leq \frac{K}{(n+1)!}.
\]
An application of the intermediate value theorem to the function $\frac{f^{(n+1)}}{(n+1)!}$ shows that there exists a number $\gamma$ between $a$ and $x$ such that

$$\frac{f^{(n+1)}(\gamma)}{(n+1)!} = \frac{R_n(x, a)}{(x-a)^{n+1}}.$$ 

Now modify this argument for the case $x \leq a$. \hfill \Box

*Alternative expression of the remainder term:* The remainder term can also be expressed by the following formula:

$$R_n(x, a) = \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-s)^n f^{(n+1)}(a + s(x-a)) \, ds.$$ 

It is obtained from $(**)_n$ by making the substitution $t = a + s(x-a)$ (so $dt = (x-a)\, ds$ and the integral from $a$ to $x$ is changed to an integral over the interval $[0, 1]$.

In Math 521 I use this form of the remainder term (which eliminates the case distinction between $a \leq x$ and $x \geq a$ in a proof above).

*Remark:* The conclusions in Theorem 2 and Theorem 3 are true under the assumption that the derivatives up to order $n+1$ exist (but $f^{(n+1)}$ is not necessarily continuous). For this version one cannot longer argue with the integral form of the remainder. See Rudin’s book for the proof.